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OVERBALANCING IN RESIDUAL-LIQUIDATION COMPUTATIONS

By Alfred S. Niles

Stanford University

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SUMMARY

The analysis of statically indeterminate structures by Southwell's relaxation method is often excessively tedious. In the present report methods are developed for the systematic design of group displacements which expedite the liquidation of residuals in certain classes of problem, and the application of these methods is illustrated by numerical examples. It is shown that when the tables of functions are available the use of the technique developed leads to a considerably large saving in computation labor and extends the field of practical application of Southwell's method. In order to facilitate the expansion of the tables of functions, the methods used in their computation are outlined in some detail.

INTRODUCTION

In recent years there has been much interest in the system of analysis originated by R. V. Southwell and termed by him the "relaxation method." This method is not a specific procedure applicable to only a limited class of problem but a general technique for attacking many classes of problem. Although it has been demonstrated that the method can be used to great advantage to analyze problems that are highly intractable to older and more conventional methods, the true limits to its field of effectiveness are still unknown, and new applications are constantly being developed. An important characteristic of the method is that unless special devices are employed, the numerical work involved in its use is likely to become impracticably tedious, and its limits of practical use are correspondingly restricted. The objective of the present paper is to present a system of applying Southwell's basic method in a manner that will often greatly reduce the amount of computation labor needed to obtain the desired result and thus to extend its range of effective use. In the present paper Southwell's relaxation method is termed the "residual-liquidation method." This work was conducted at Stanford University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

[1]Complete tabular data and application of this method to several panel problems are contained in an unpublished report, "Overbalancing in Residual Liquidation Analyses" by Alfred S. Niles, which is available for reference or loan in the Division of Research Information, National Advisory Committee for Aeronautics, Washington, D. C.

RESIDUAL-LIQUIDATION METHOD

Basic Procedure

Before considering the present modifications it is advisable to present a brief outline of the basic method and to define a number of technical terms according to the meanings with which they are used throughout this report. Before a structure can be analyzed by residual liquidation, it is necessary to construct an "operations table" or equivalent "operations diagrams." For this purpose the structure is divided into a number of elements connected at suitably chosen points, which may be called "joints" or "stations."

Each joint is then assumed to be displaced with respect to the remainder through a known distance, and from the resulting strains the internal forces that would be developed against the joints are computed. One of these forces will act at the displaced joint, along the line of the displacement, but in the opposite direction. This is called the "resistance" at the displaced joint. The other forces acting against the joints are the "carry-over forces" from the displaced joint to the joints at which they act. The operations table gives the resistances and carry-overs associated with the various joint displacements that are likely to be of interest. In order to be complete it should also include the magnitudes of the displacements involved. If the displacements are 1 unit of distance in magnitude the resistances and carry-overs are designated "unit." If they are of such magnitude that the resistances at the displaced joints are equal in magnitude to a unit of force, the displacements and carry-over forces are designated "basic." In practice an operations table may be limited to unit or basic values, but it is sometimes convenient to include both. The ratio of a carry-over force to the associated resistance is a "carry-over factor." The resultant of the system composed of the resistance and carry-over forces associated with a single joint displacement must be zero. An operations diagram differs from an operations table only in the manner of presenting the data.

After the operations table has been constructed, a structure is ready for analysis by residual liquidation. The first step is to assume the presence at each joint of imaginary rigid "constraints" capable of developing the reactions that may be needed to maintain the joint in equilibrium. Then the known external loads and reactions are assumed to be imposed at the appropriate joints without causing any joint displacements. At this stage each of these forces must be considered as an "unbalanced force"; that is, it is not balanced by any forces in the actual structure since that structure is still assumed to be without strain. Its elements therefore develop no resisting forces and the constraints must be relied upon to maintain equilibrium. If, now, the joint at which the largest unbalanced force acts should be displaced along the line of action of that force, an internal force will be developed which will reduce that which the constraints must provide. From the operations table one can determine the amount of displacement required to reduce to zero the force that must be provided by the constraints at the displaced joint. If that displacement is assumed, the displaced joint is said to be "balanced."

In general the balancing of a joint does not eliminate the unbalanced force that acted there prior to the balancing operation. A statically equivalent group of forces is distributed among or carried over to the adjacent joints. The result of displacing a joint is therefore best described as a "force transfer" or "force transformation." When the force carried over to a joint is equal and opposite to an unbalanced force already acting at that joint, the two neutralize each other and may properly be considered to have been "liquidated." In any case the algebraic sum of the preexisting unbalanced force at a joint and that carried over to it as a result of balancing an adjacent joint will now be the unbalanced force at that joint which must be held in equilibrium by the constraints. For convenience it is called the "residual" at the joint in question.

Normally the individual forces carried over to adjacent joints when a joint is balanced are smaller than the resistance developed at the balanced joint. Theoretically it is possible, by successively balancing the joint subjected to the largest residual, to reduce the magnitude of that largest residual until all of the original unbalanced forces have been so transferred that they have come into opposition with their equilibrants and have been liquidated. This condition would be indicated by the fact that the residual at every joint equals zero. Actually, however, except in the simplest structures, to reach that stage would require an infinite number of steps. In practice therefore the analysis is stopped when the residuals have been reduced to negligible magnitudes. The record of original unbalanced forces, assumed joint displacements, resulting resistances, carry-over forces, and residuals is a "liquidation table."

If the liquidation of the residuals is carried out in the manner just described, the number of steps required is likely to be excessive. This is due primarily to the fact that after a joint has been balanced it is thrown out of balance by carry-over forces developed by the balancing of the adjacent joints. One method of reducing the required number of steps is to employ "group displacements" in which several joints are assumed to be displaced simultaneously. If the group displacement is one in which there is no relative movement between the joints of the moving group it is known as a "block displacement." The resistances and carry-over forces associated with any given group displacement can easily be determined by applying the method of superposition to the data in the operations table and can be included in that table. The liquidation of residuals can often be greatly expedited by the use of suitable group displacements.

Theory of Overbalancing

Another line of attack is to overbalance the joints. Instead of assuming merely the displacement needed to balance a joint under the assumption that the adjacent joints are not displaced, a larger displacement is assumed. One result is that the unbalanced force or residual at the joint is replaced by a residual in the opposite direction. Then, when the adjacent joints are assumed displaced, the carry-over forces will tend to liquidate that residual. If the correct displacements were assumed, each joint would need to be displaced only once in order to liquidate all the unbalanced forces. This ideal is seldom attainable but it is often possible to make such good estimates regarding the necessary displacements that the liquidation will be greatly expedited.

When a single joint is displaced the resulting force transfer affects only the residuals at the displaced joint and those adjacent to it. When several joints are displaced simultaneously the changes in the residuals due to the resulting force transfers may fall into any one of a number of "transfer patterns." Thus a single force may be transferred to one or more joints at various distances from its original position, a single force may be distributed among other joints in various proportions, a group of forces may all be transferred to a single joint, and so forth. It is possible to construct an "overbalance table" or "overbalance diagram" showing the relative displacements required and resistances developed at the joints involved, in order to transfer forces in a given pattern. With tables for the more useful patterns available as guides, it is possible to overbalance the joints so as to liquidate the residuals quite efficiently. In some cases the original unbalanced loads can be carried directly to the joints where they come into opposition to their equilibrants. Usually, however, it is necessary to carry through several cycles of residual liquidation, though many fewer than if no advantage were taken of the possibilities of overbalancing.

An overbalance table for a given transfer pattern should include the "original forces" supposed to exist prior to the transfer, the "final forces" which replace them, and the "resistances" which would be developed by each displacement of the group if it were the only one to take place. In this report, the term "the resistance" is consistently used for the resistance that would be developed at a joint, parallel to its displacement, if its assumed displacement were the only one to be considered. The resultant force parallel to the displacement of a joint resulting from an assumed simultaneous displacement of that and other joints is termed the "net force" at that joint. The final force parallel to the displacement of a joint is the algebraic sum of the original force, if any, and the net force. The tables constructed for general use are called "characteristic overbalance tables" and each is built up around the transfer of a "characteristic load." This may be a single original force which is to be replaced by one or more final forces. It may also be a single final force which is to replace a group of original forces. It may even be the sum of a group of original forces that is to be replaced by another group of final forces. It would have been possible to construct the characteristic overbalance tables of this report on the basis of a characteristic force of unity in every case, but that would have been inconvenient. It is preferable to use characteristic forces of magnitudes appropriate to the individual transfer patterns and give the characteristic force in each overbalance table. The characteristic load, resistances, and so forth, appearing in the overbalance table for a given transfer pattern are the "overbalance factors" for that pattern.

Since it is only accidentally that it will be desired to transfer a "specific load" exactly equal to the characteristic load of a characteristic overbalance table, the actual force to be transferred would be divided by the characteristic force to obtain the "adjustment ratio." Multiplication of the values in the characteristic overbalance table by the adjustment ratio then produces the data for a "specific overbalance table" or "transfer record" for the load under consideration. The resistances shown in the characteristic overbalance tables are called "characteristic resistances," and those in a specific overbalance table are "specific resistances." At times it is convenient to include the carry-over forces in an overbalance table or diagram. In this case, it is desirable to designate them either characteristic or specific, according to the type of table or diagram with which they are connected.

Overbalance tables and diagrams could be devised to fit structures composed of elements of varied size, but each such structure would require its own set of tables and they would be valid only for that structure and others that might be geometrically similar. Such tables will be found in the analysis of the structures in question. The only overbalance tables of general application are those designed for "uniform structures" composed of a number of identical elements, and most of those discussed in this report are of this character. A representative structure of this type is a flat sheet reinforced by equal-sized, uniformly spaced, longitudinal stringers and equal-sized, uniformly spaced, transverse ribs. Here the basic element is the panel bounded by adjacent ribs and stringers. The natural locations for the joints at which the constraints are assumed to act are the intersections of ribs and stiffeners and it is to be noted that these joints are naturally grouped into "sequences" along the individual ribs and stringers. In this example each joint is a member of two different sequences, and the sequences may be divided into two families. In general the joints of a uniform structure fall naturally into such sequences, and the sequences may be grouped into one or more families. With some structures all joints may fall into a single sequence.

In general the joints of a sequence lie along a straight line. If one joint is displaced along the line of a sequence of which it is a member, the sum of the carry-over forces to the adjacent joints of the sequence may or may not be equal and opposite to the resistance developed at the displaced joint. The ratio of this sum of the carry-over forces to the resistance is the "transfer factor" for the sequence and depends on the physical properties of the structure. If the transfer factor is other than unity, the difference represents "leakage" or forces transferred from the sequence in question to parallel sequences.

Although uniform structures with all joints in a single sequence and with a transfer factor of unity are relatively uncommon, they exist in sufficient number to justify the construction of overbalance tables for them. Such tables are easier to devise than those for structures with smaller transfer factors and it is advisable to discuss them in some detail before taking up the more complex problem of structures with transfer factors less than unity.

CONSTRUCTION OF OVERBALANCE TABLES

Transfer Factor k Equal to Unity

The method used to construct a characteristic overbalance table can be best explained by carrying out an illustrative example in detail. Let the example be the transfer of a force from an intermediate joint of a sequence to the end joints, the latter being assumed not displaced. In order to make the problem more specific, assume that the sequence consists of nine joints numbered serially from 0 to 8 and that the force to be transferred to joints 0 to 8 is located at joint 5. This situation is shown diagrammatically in figure 1. In this case it will be noted that the original load system is the single force of 8 units applied at joint 5 and the selected final force system is composed of 3 units at joint 9 and 5 units at joint 8. These values were chosen because it could easily be determined from the principle of consistent deformations that, if the structure were a uniform bar subjected to an axial load at joint 5 and the ends were fixed, three-eighths of that load would be carried to joint 0 and five-eighths to joint 8.

If any joint i is displaced along the direction of the sequence, a resistance R_i will be developed that will act against the constraints at joint i, and if the transfer factor k is equal to 1, forces of $-R_i/2$ will be imposed on the constraints at joints $i - 1$ and $i + 1$. Figure 2 shows figure 1 with added rows in which these resistances and carry-over forces are given. Since the final force at each joint is known, the values of R_i can be found by solving the simultaneous equations expressing the relation of each final force to the original force, resistance, and carry-over forces at the joint where it acts. In figure 2 the resistances are shown in the R-row and the carry-overs in the two CO-rows, one for forces carried over from joint $i - 1$ and the other for forces carried over from joint $i + 1$.

At joint 0, $-0.5\ R_1 = 3$, whence $R_1 = -6$

At joint 1, $-0.5\ R_2 + R_1 = 0$, whence $R_2 = -12$

At joint 2, $-0.5\ R_1 + R_2 - 0.5\ R_3 = 0$, whence $R_3 = -18$

Similarly the other values of R_i are found to be: $R_4 = -24$, $R_5 = -30$, $R_6 = -20$, and $R_7 = -10$.

In practice, resistances and carry-overs can usually be found more easily by a slightly different procedure. First enter the original and final forces, and allot spaces on the diagram for the carry-over forces and resistances. It should be obvious that the only source of a final force at joint 0 is the carry-over from joint 1 and that that carry-over force must be minus one-half the resistance developed at joint 1. Therefore enter 3 as one carry-over force at joint 0 and −6 as the resistance at joint 1. Since the displacement of joint 1 that produces a carry-over force of 3 at joint 0 also develops a carry-over force of 3 at joint 2, the latter force should be entered in the other carry-over row at joint 2.

The final force at joint 1 is to be zero. Since the original force at that joint is zero and no force is carried over to joint 1 from joint 0 (since joint 0 is assumed undisplaced), the carry-over from joint 2 to joint 1 must be equal and opposite to the resistance for joint 1. Therefore enter 6 as carry-overs at joints 1 and 3 and -12 as the resistance for joint 2. At this stage the carry-over force at joint 2 due to displacement of joint 3 has not been determined, but from the resistance and the carry-over force that has been found it is clear that if the original and final forces at that joint are to be zero the unknown carry-over must be 9. Therefore 9 is entered as the two carry-overs from joint 3 and −18 as the resistance for that joint. Similar computations based on conditions at joints 3 and 4 justify entering 12 as the carry-overs from and −24 as the resistance for joint 4 and 15 as the carry-overs from and −30 as the resistance for joint 5.

When computing the required carry-over forces from joint 6 from conditions at joint 5 the original force of 8 must be considered. The desired carry-over is therefore 30 − 12 − 8 = 10 and the resistance for joint 6 is −20. On continuing by the same system, the resistance for joint 7 is −10 and the associated carry-over forces are each 5. This carry-over force is the only force developed at joint 8 and thus becomes the final force at that joint. Since it agrees with the assumed final force, the overbalance table is checked.

If desired, almost the same system of computation could have been used, except that work would have proceeded from both end joints to the intermediate loaded joint. Then the resistance for the originally loaded joint would be computed from two sources, and equality of the independently computed results would prove the correctness of the table. In this example the overbalance table was constructed for a characteristic load of 8 units. This was done primarily to avoid the use of fractions.

If the spaces between the joints of a sequence are designated "panels" or "bays," the overbalance diagram of figure 3 is that for a special case of the general family of transfer patterns in which a characteristic load of $m + n$ pounds imposed at joint n of a sequence of $m + n$ bays is transferred to the end joints 0 and $m + n$. In this case $n = 5$ and $m = 3$. Separate overbalance tables or diagrams could be constructed for each likely combination of values for m and n. When the transfer factor is unity, however, it is preferable to develop formulas for the resistances and construct, for the purpose of this report, a single illustrative example. Assume, for example, that a characteristic load CL acting at joint n of a sequence of $m + n$ bays is to be transferred to the stationary end joints 0 and $n + m$. The most convenient value for CL will be $m + n$. The final force at joint 0 will be m and that at joint $m + n$ will be n. For joints between 0 and n the resistance will be $R_i = -2im$, where i is the number of the joint. Between joints n and $m + n$ the resistances can be obtained from the relation $R_i = -2(m + n - i)n$. The same result can be obtained more conveniently by interchanging m and n and counting i from joint $m + n$. The fact that the resistances of figure 3 could have been obtained from these relations can be easily checked.

When the original load is imposed at the center joint of a sequence, $m = n$. One-half the original load will be carried to each end joint, and the required resistances R_i will each be equal to $-i$, where i is the number of bays from the joint in question to the nearest end joint. The value of CL for this transfer pattern would be 1.

The basic methods used in the preceding discussion for determining resistances for transferring a single load to the fixed ends of a sequence of joints are equally applicable to other types of force transfer when the transfer factor is unity. It is not necessary to discuss each type of transfer pattern in detail, but appendix A includes discussions of sample overbalance tables and associated formulas for the resistances for a number of transfer patterns. It is believed that this material is given in such form that the diagram or table required for other values of m and n can be constructed without difficulty.

Transfer Factor k Not Equal to Unity

When the transfer factor k has a value other than unity the determination of required resistances becomes somewhat more complex, though its essential nature remains unchanged. In general the first step is to set up an overbalance diagram similar to figure 2. Figure 4 is such a diagram for the transfer of a concentrated force from the center to the end joints of a span of eight bays, the end joints being assumed fixed. The important differences between figures 2 and 4 are:

1. Instead of listing numerical values for both the original and the final forces, only the final forces are given numerical values. The original force is indicated by the abbreviation CL for characteristic load. The fact that the final forces at the end joints would be equal follows the symmetry of the transfer pattern. The numerical value of unity for each of those forces was chosen arbitrarily on the basis of convenience.

2. The various carry-over forces are listed as minus $k/2$ times the resistances R of the adjacent joints, instead of as minus one-half those resistances. Obviously $\frac{k}{2} = \frac{1}{2}$ when $k = 1$.

By using the diagram of figure 4 as a guide, the resistances associated with the transfer pattern under consideration can be determined in the same manner as those shown in figure 3. If $k = 0.9$, $\frac{2}{k} = 2.22222$. At joint 0 the carry-over force from joint 1, $-0.5kR_1$, must be equal to 1.00000, whence $R_1 = -2.22222$. Then at joint 1, since the original force, final force, and carry-over from joint 0 are all zero, $-0.5kR_2 + R_1 = 0$. On using the value just computed for R_1, $R2$ must be $-2.22222 \times 2.22222 = -4.93826$. All the forces acting at joint 2 except R_3 have now been determined, so the latter must be

$$R_3 = 2.22222\ (-4.93826 + 1.00000) = -8.75168$$

The resistances for all joints up to and including that at which the original load is applied can be determined in the same manner. The work is facilitated by arranging the computations as shown in table 1, where the method of computation is indicated in the column headings. The first five columns of this table apply to the computation of the desired resistances. At joint 4, the value of CO2 is that which would be applicable if the original load were applied at a joint farther along the sequence. If, however, joint 4 is at the center of the sequence and is the point of application of the original force, from the symmetry of the loading the actual value of the carry-over force from joint 5 would be the same as that from joint 3, or 3.93826. The net force developed at joint 4 would then be $-14.50990 + 2 \times 3.93826 = -6.63338$. In other words the displacements which would produce the resistances given in column (3) would liquidate 6.63338 pounds at joint 4, of which 2.00000 pounds would be equally divided between the end joints, and the remainder would be transferred to joints outside the sequence. The characteristic load for this type of transfer pattern for a sequence of eight bays and a transfer factor $k = 0.90$ is therefore 6.63338. This value could also have been obtained by subtracting the value 3.93826 of column (4) from the value 10.57164 of column (5).

If the sequence had been only six bays in length and the original force had been applied at joint 3, that force would necessarily have been 4.30724 pounds, as can be determined from a parallel computation. Similarly obtained values for the original force or characteristic load for sequences of two and four panels are also given in column (6).

It can easily be seen that table 1 can be extended to give the values for the resistances and characteristic loads for any desired even number of bays, and the values obtained in this manner for several values of transfer factor k are given in table B-1.*[2] For any specific number of bays n, it would be possible to divide the pertinent values of resistances from column (3) by the characteristic load from the $\frac{n}{2}$ -row of column (6).

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[2]Tables B-1 to B-9 referred to in the present paper are designated with an asterisk to indicate that they are contained in the complete report, which is available for reference or loan at the NACA.

The resistances obtained in this manner would apply to the liquidation of a unity load applied at the center joint. If that were done, however, it would be necessary to make a separate table of resistances for each value of $n/2$. It is preferable to have a single set of resistances and modify the characteristic load in accordance with the length of the sequence.

When computing resistances for the same transfer pattern as that of table 1, but with the transfer factor of unity, it was possible to use the conditions at the center joint to check the computations, since the original force at that joint was already known. When the transfer factor is not unity those conditions are needed to compute the value of the original force that corresponds to the resistances obtained. It is to be noted that this original force is considerably larger than the sum of the unit final forces at the end joints. From table 1 it can be seen that this difference increases rapidly with an increase in the sequence length. It also increases with the difference between the transfer factor and unity. This "leakage" of force from a sequence to adjoining sequences is one of the chief reasons why the liquidation of residuals may require numerous cycles for its accomplishment.

It is to be noted that the title of table 1 is not a complete description of the transfer pattern to which it applies. It mentions only those original and final forces which are parallel to the assumed displacements and which also act at joints of the sequence under consideration. When the transfer factor is not unity there will also be final forces parallel to the assumed displacements at joints that are not in the sequence under consideration, that is, leakage. With any value of transfer factor there may also be carry-over and final forces normal to the assumed displacements. Although such leakage and normal forces actually constitute part of the transfer pattern, it is convenient to omit mention of them in order to obtain a concise designation. The practice of limiting the designation of a transfer pattern to the changes in those forces on the joints of the sequence under consideration which act parallel to the assumed displacements is followed throughout this report.

Essentially the same method of attack as that described in the preceding discussion can be used to obtain such overbalance factors as resistances and characteristic loads for other types of transfer pattern likely to be of interest. The overbalance factors computed for transfer factors other than unity are given in tables B-1 to B-9.* The methods of computation used are outlined in appendix B. The types of transfer pattern covered by the tables include the following:

1. Transfer of single force from center to fixed end joints

2. Transfer of single force from any intermediate joint to fixed end joints

3. Transfer of single force to a single fixed end joint

4. Uniform distribution of a single load applied at one end of a sequence, the other end joint assumed fixed

5. Transfer of uniformly distributed loading to end joints without net leakage or change in sequence length

6. Transfer of uniformly distributed loading to fixed end joints

7. Uniform distribution of single force applied at any joint without net leakage

8. Uniform distribution of single load applied at one end of a sequence, that end or some intermediate joint assumed fixed

9. Mutual liquidation of equal but opposite uniformly varying loadings

In appendix B rules are also given for handling some other types of force transfer and the development of these rules is outlined.

This material does not cover as great a range of possible transfer patterns as are included in the rules for a transfer factor of unity but includes those most likely to be of practical importance. In most of the patterns considered, one or two of the joints is assumed to be fixed, that is, to have no displacement. Whenever this assumption is made there will be such leakage that the algebraic sum of the final forces will not equal that of the original forces. If, however, either the original or the final loading is uniformly distributed and all of the joints are assumed displaced, it is possible to proportion the displacements so that the algebraic sum of the leakage forces will be zero. The patterns in which this is accomplished are those described as being without net leakage. In the numerical examples it is shown that these transfer patterns are often of great value in computing.

APPLICATION OF OVERBALANCE FACTORS TO THREE-BAY FLAT PANEL

The practical details of the use of overbalance factors and the degree to which they expedite residual-liquidation computations can be best illustrated by numerical examples. In order to have these examples serve the latter function, several of them apply to structures for which residual-liquidation analyses have been published. Some of them, in fact, are alternative computations for the structures analysed by Dr. Hoff and his associates in references 1 and 2. The use of the structures analysed in these reports has the additional advantage that it makes the computation of influence coefficients for the present report unnecessary, and considerable use can be made of the operations tables contained therein.

The first example to be considered will be the structure investigated on pages 17 to 22 of reference 1. This was a flat panel reinforced by four stringers and four transverse ribs. The lower end of each stringer was subjected to a downward load of 60.0 pounds and the panel was supported by an upward force of 120.0 pounds at each upper corner. A schematic drawing of one-half of the panel and its loading is shown in figure 5, which is a copy of figure 22 of reference 1. Because of symmetry only one-half of the panel need be investigated, and forces acting normal to the stringers may be neglected.

The first steps in analyzing this panel by overbalancing are the same as those employed by Hoff. First, it is necessary to compute the influence coefficients from the geometry of the structure and to use those coefficients to construct an operations table like that on page 18 of reference 1. With only eight joints and one direction of displacement per joint to be considered, it is convenient to record the operations data in the tabular form. An alternative is to use diagrams like figures 6 and 7. In figure 6 each square represents a joint. The square in the middle of the left-hand column represents the joint which is assumed to be displaced 1 unit in the positive direction as is indicated by the notation $v = 1.00$ entered in the square. Since unit displacement of an intermediate joint of stringer AEJN develops a resistance of −101.6 pounds at that joint, the value −101.6 is also entered in that square. The same displacement of an intermediate joint develops carry-over forces of 46.8 pounds at each of the adjacent joints in the same stringer, so 46.8 is entered in the squares above and below that first mentioned. The squares in the right-hand column represent joints in the stringer to the right of that in which the displacement is assumed. Since carry-over forces of 2.0 pounds would be developed at the joints just above and below and 4.0 pounds at the joint on the same level as the one displaced, those values are entered in the appropriate squares. If, however, the displaced joint is one at the end of the sequence, the resistances and carry-over forces are modified. Thus if the square in the middle of the left-hand column should represent joint A at the upper end of the stringer, unit displacement would develop a resistance of only −50.8 pounds, and the carry-over to the opposite joint of the adjacent stringer would be only 2.0 instead of 4.0 pounds. There would be no carry-overs to joints represented by the two upper squares, while those to the lower joints would be unchanged. The forces resulting from the unit displacement of an end joint are listed on the diagram in parentheses. Although these values are entered in locations that imply the joint displaced was at the upper end of the sequence, there should be no difficulty in using them in connection with an assumed movement of a lower end joint.

For figure 6, $k = 0.921$. Here k is the transfer factor and is numerically equal to $2 \times 46.8/101.6$ or $46.8/50.8$. It is the measure of the amount of resistance developed at the displaced joint that is carried over to other joints of the sequence. In figure 6 a unit block displacement of the stringer would develop a resistance of −8.0 pounds at each intermediate joint. In such a movement the resistance developed at each end joint would be just one-half as great, or −4.0 pounds. Since the resistance developed at an end joint by a block displacement is always one-half that developed at an intermediate joint in a uniform structure, it is not necessary to give the resistances developed at both kinds of joint.

The values given in figure 6 do not apply to stringer BFKO because of the larger effective area of the member. The appropriate values are shown in figure 7. In this diagram the joints of the stringer in which the displacement is assumed are represented by the right-hand column of squares. That is convenient in connection with the problem under study. It may be noted that almost all values appearing in figures 6 and 7 are taken from the operations table of page 17 of reference 1. The only exceptions are the values of the transfer factors k, which were computed from the other listed values.

After the operations table or diagrams have been set up, the known external forces should be entered in the liquidation table, as has been done in row 1 of table 2. From this table it is obvious that the net unbalanced force on stringer AEJN is 60.0 pounds and that on BFKO is −60.0 pounds. The panel as a whole may be considered to be "in general balance," though neither of the individual stringers is in that condition. In a problem of this type it is desirable as a general rule to get each stringer into general balance as soon as possible and to restore it to that condition as soon as possible after it may be unbalanced by assumed displacements. From the operations diagrams it can be seen that since a unit block displacement of a stringer will develop a resistance of −8.0 pounds at each intermediate joint and the stringer length is three bays, the total resistance developed would be $-3 \times 8 = -24.0$ pounds. A block displacement of $60/24 = 2.500$ units will develop resistances that will eliminate the unbalance of stringer AEJN and carry-over forces that would eliminate the general unbalance of the other stringer. The magnitude of this assumed displacement and the resulting resistances and carry-over forces are listed in row 2 of the liquidation table and the resulting residuals, in row 3.

Each stringer is now in general balance, but each individual joint is subjected to a larger residual than is considered desirable. Since the arithmetic sum of the residuals in the left-hand stringer is the larger, that member is attacked first. The residuals may be treated as a combination of a uniformly distributed load of −60 pounds, a single load of −60 pounds at joint N, and a single load of 120 pounds at joint A. The first of these is already uniformly distributed, and the other two may be uniformly distributed without net leakage with the aid of the overbalance factors listed in table B-7.* For this purpose it is desirable to "design a group displacement" in the manner illustrated by table 3. In the first column each joint of the sequence is entered as many times as there are joint loads to be distributed. In the second column are entered the original joint loads, one in each section of the table, in the row for the joints at which they act. The characteristic loads of the third column are taken from table B-7.* In this case, since both joints at which there are original loads are at the ends of the sequence, the two characteristic loads are of the same size. The numerical values are taken from the section of table B-7* which applies to the distribution of a single force imposed at joint O of a sequence of three bays. Obviously either joint A or joint N may be considered as joint O. The figures used are from the column for $k = 0.92$, that value of k being the closest to the actual value of 0.921 for which values are given. It would be possible, but hardly desirable, to interpolate, particularly when the number of bays in the sequence is so small. It is better to use the given values and go through another step of liquidation if necessary.

The adjustment ratios of column (4) are found by dividing the original forces of column (2) by the corresponding characteristic loads of column (3), due account being taken of signs. The characteristic resistances of column (5) are the resistances listed in the same portion of table B-7* as that from which the characteristic loads were obtained. In the first part of the table the joints are assumed numbered from A to N and in the second part, from N to A. The adjustment ratio for each part of the table is multiplied by the corresponding values of the characteristic resistance to find the specific resistance that should be developed at each joint to obtain uniform distribution of the original force involved. The results are shown in columns (6) to (9), a separate column being provided for each joint. The total resistances to be developed at the joints are found by adding the values in columns (6) to (9), and they are entered in the R-row. The values of R/v in the next row are taken from the operations diagram, and then the numerical values for the required joint displacements can be easily computed and entered in the v-row.

Once the group displacement has been designed in this manner, return to the liquidation table and enter in the displacement column the individual displacements just found, one to a row. Then the values for resistances and carry-over forces given in the operations diagram or in the operations table are multiplied by the respective displacements to fill out rows 4 to 7 of table 2. In this step some time may be saved by entering the total resistances to be developed at each joint, as computed in table 3, directly in table 2. There is no particular point in dividing each of these values by the appropriate resistance per unit displacement, losing the slide-rule setting, and then multiplying the result by the previous divisor to get a slightly different result.

Adding the forces given in rows 3 to 7 of table 2 gives the residuals shown in row 8. Those for joints A, E, J, and N are negligible; this shows that the group displacement designed in table 3 is effective in liquidating the forces on that group of joints. Because of leakage, the residuals on stringer BFKO no longer present the simple pattern of a uniformly distributed load plus a single load at joint O. Each joint load, however, may be treated in the same manner as the single loads at joints A and N were treated in table 3. Table 4 records the computations for the design of a group displacement to liquidate these joint loads. It was constructed in the same manner as table 3, except that the characteristic loads and resistances for the distribution of original forces at joints F and K are taken from the section of table B-7* pertaining to the distribution of a single force applied at joint 1, instead of joint O, of a three-bay sequence. The factors used are for $k = 0.92$, as in table 3, although the actual value of k for stringer BFKO is about 0.928.

The displacements computed in table 4 and the associated resistances and carry-over forces are entered in rows 9 to 12 of table 2, and the corresponding residuals are shown in row 13. The liquidation of the residuals on stringer BFKO after this third displacement is not as complete as that of the forces on AEJN after the second. This is probably due primarily to the greater difference between the actual transfer factor for the stringer and that assumed in designing the group displacement. Even so these residuals are quite small. The residuals on AEJN after the third displacement are of appreciable size, owing to forces carried over from BFKO in that displacement. If great precision of results were not required, the analysis might be terminated at this point, but in the example a fourth displacement was designed to obtain more complete liquidation. The computations for the design of the group displacement are recorded in table 5 and the results are used to obtain the values shown in rows 14 to 18 of the liquidation table. At the end of this fourth cycle the maximum residual is only 0.7 pound and might well be neglected. Of course it would be possible to design another group displacement to eliminate the residuals of row 18, but it would probably be better to add up the displacements of each joint and construct a check table like that of reference 1. If the check table resulted in residuals of larger size than is considered desirable, they could be liquidated by additional group displacements designed in the same manner as those of tables 4 and 5.

In carrying out the computations outlined, there are several chances to make independent checks of the work. In table 2 the algebraic sum of the forces given in each row should be zero. In the same table after each group displacement designed to effect a force transfer without leakage, the algebraic sum of the residuals for the joints in each sequence should be zero. In the tables in which the group displacements are designed, the algebraic sum of the forces in columns (6) to (9) associated with each original force of column (2) should be equal to zero, and the algebraic sum of the total resistances of those columns should be zero. In the same tables the values for the characteristic resistances in column (5) should be symmetrical about the middle of the column. These criterions can be used if desired to make minor adjustments in the computations.

The total joint displacements of table 2 are not the same as those recorded by Hoff because the point of implied zero displacement was different. Table 6 gives the total displacements of table 2, the corresponding displacements as found by Hoff, and those of table 2 minus the value for v_F. Since Hoff assumed that $v_F = 0$, it is the last mentioned set of displacements which should be compared with those of table 2. It is obvious that the agreement is very good. The largest discrepancies are at joints N and O where Hoff's residuals were largest. Since a displacement of 0.04 unit of joint N or O would develop a resistance of only about $53 \times 0.04 = 2.06$ pounds, the differences between Hoff's values and those of this report are not important. It is believed that the displacements of the present report are the more accurate since forces were computed to the nearest one-tenth of a pound instead of to the nearest pound, and the final residuals happen to be smaller.

CONCLUDING REMARKS

In attacking a problem by the method of residual liquidation, it is necessary to keep in mind that the result of any displacement of either a single joint or a group of joints is to transfer residuals from one set of locations to another and that the residuals after the displacements must be statically equivalent to those preceding the displacements. Therefore it is desirable to arrange the problem in such a manner that the summations of forces and moments are zero at the start and to make occasional checks to make sure that those relations still hold. Sometimes, because of the lack of knowledge regarding the reactions, this will not be possible, but such a condition can easily be taken care of by substituting the proper constants for the zeros of the usual equations and always taking the summation of moments about the same axis.

In practical work errors due to the use of a limited number of significant figures are almost certain to be introduced into the computations. There is also the possibility of making mistakes producing relatively large errors. As soon as the residuals have been cut down to small magnitudes it is normally desirable to add the displacements that have been assumed and use the results as a single group displacement in a new liquidation or check table. The residuals of this check table are more reliable than those of the original liquidation table, and if they are of such size that additional liquidation is desirable, it is the former that should be used in the computations. In general it is probably desirable to construct such a check table just before making the step that is hoped to be the final step of liquidation. Then, if that step results in negligible residuals after small displacements are added to those of the first check, it would not be necessary to construct another check table for the total displacements. If, however, several steps of liquidation are required after the construction of the first check table, a second check table based on the final displacements would be in order.

None of the structures investigated in connection with this report included a very large number of joints. Therefore it was convenient to enter each joint or stringer displacement and the associated resistances, carry-over and leakage forces in a separate row of the liquidation table. With more extensive structures that would not be true. Then it would be more convenient if the forces developed in each step were entered on a separate "transfer record," which would have columns for only those joints at which such forces would appear. In many cases that would be only a fraction of the total. The algebraic sum of the forces developed at each joint would then be entered in a single row of the main liquidation table, which would thus be greatly condensed.

It would be desirable to make a more complete study of the effect of using an approximate rather than the precise value of the transfer factor k. Such study of this problem as was included in this investigation indicated that the importance of using a precise value for k increased rather rapidly with the number of joints in the sequence. Also, as the number of joints increases the individual resistances increase rapidly in size. This effect is at least partly neutralized, however, by the fact that the characteristic loads similarly increase with the number of joints per sequence. If both sets of figures were to increase at the same rate, the effects would tend to neutralize each other. A similar problem is the effect of deviation of the transfer factor k from unity. In general it is found that the greater the difference between k and unity the larger will be both the resistances and the characteristic loads.

The chief factor which tends to prolong the process of liquidation in practical problems is "leakage." It appears relatively simple to put each stringer in general balance or to transfer part of a load from one part of a sequence to another, but much of that load will be transferred to adjacent stringers and reappear when those adjacent stringers are considered. It would be desirable to investigate the possibilities of obtaining overbalance factors for "two-dimensional sequences" or "matrices" of joints. Such an investigation might prove to be a useful extension of the investigation.

Although the use of overbalance factors does not eliminate all the problems connected with the residual-liquidation method originated by Southwell, the material in this report demonstrates that it is a useful adjunct to that method and facilitates the solution of certain types of problem, particularly some of those associated with panel analysis.

Guggenheim Aeronautic Laboratory
Stanford University
Stanford University, Calif., March 17, 1945

APPENDIX A

OVERBALANCE FACTORS FOR A TRANSFER FACTOR OF UNITY

The rules for computing the overbalance factors associated with a transfer factor of unity are expressed by means of sample overbalance tables (tables 7 to 25) and formulas. In each sample table the following quantities are given for each joint of the sequence (except that for some symmetrical loadings only a part of the sequence is covered): original force OF, resistance R, carry-over force from preceding joint CO1, carry-over force from following joint CO2, and final force FF. The numbers of the joints assumed not to move are indicated by the letter F in the joint column. The type of transfer pattern is described in the table caption and can easily be verified by inspection of the joint, OF-, and FF-columns. Accompanying each sample table is a bracketed head note for assistance in constructing a table for the same type of transfer pattern but a different number of joints. In general, these head notes give the following information. First are given the number of bays between joints in the whole sequence for which the sample table was constructed, or the number of bays in each of its major segments. This is followed by the general rule for computing the characteristic load for a transfer of the type under consideration. The third item gives the shape of the curve, termed the "displacement curve," that would result from plotting the displacements required to develop the given resistances against joint positions. Finally general formulas for the resistances are given.

At the end joint of a sequence the ratio of resistance to displacement is only one-half that at an intermediate joint. Therefore a curve of joint resistances plotted against joint locations would not have the same shape as the corresponding displacement curve, except for transfer patterns for which the end joints are not displaced. The shape of a curve of resistances, can, however, be easily determined from the corresponding displacement curve by halving the end ordinates of the latter. Often it is more convenient to determine the desired resistances from the shape of the pertinent displacement curve, with the end ordinates halved, than to use the formulas. In any case knowledge of the proper shape of the displacement curve is of value for checking the resistances.

In many of the transfer patterns considered, either the original or the final load is considered to be uniformly distributed. In each of these cases it is to be assumed that the loads of the end portions are only one-half the size they would otherwise be. In other words the loads are uniformly distributed between bays or panels rather than between joints.

Although only one sample overbalance table is given for each transfer pattern, it should be easy to construct similar tables for other sequence lengths. In fact when the transfer factor is unity it is a simple matter to construct an overbalance table for any desired transfer pattern, once the values of the original and final forces and the location of the fixed joint are known. The method to be used is explained in connection with the construction of table 3. When only one joint of a sequence is assumed fixed and the transfer factor is unity, an overbalance table can be constructed for any transfer pattern in which the sum of the final forces is equal to the sum of the original forces. When, however, two joints of the sequence are assumed fixed, the sum of the moments of the final forces about any joint must also equal the sum of the moments of the original forces about that joint. In making this computation, the forces should be assumed to act at an angle other than zero to the direction of the sequence.

APPENDIX B

METHODS FOR COMPUTING OVERBALANCE FACTORS FOR TRANSFER FACTOR NOT UNITY

In the text the method of computing resistances and characteristic loads when the transfer factor is other than unity is described for one case, that of the transfer of a single force from the center joint of a sequence to fixed end joints. The same basic method is used for other types of transfer pattern, but each has its special features that must be taken into account. It is therefore desirable to outline the procedures used for obtaining the other values given in tables B-1 to B-9.* A good understanding of these procedures will assist in the sound use of the overbalancing procedure and in the computation of additional overbalance factors if that should seem desirable.

Transfer of Single Force from Any Intermediate Joint to Two Fixed End Joints

When a single force is to be transferred to two fixed end joints from a joint that is not at the center of the sequence, the procedure for computing resistances and the characteristic load differs in some details from that for the symmetrical case. One difficulty is that not only are the ratios of the original force to the final forces at the end joints unknown, but the ratio of one final force to the other is also unknown. They are inversely proportional to the distances from the location of the original force to the end joints only when the transfer factor is unity. The procedure for handling the case of a single force imposed at joint n of a sequence of $m + n$ bays is as follows. Assume the final force at joint 0 to be 1.00000. The desired resistances for joints 0 to n can then be taken from table B-1.* Thus if $n = 4$, $m = 3$, and $k = 0.90$, $R_4 = -14.50990$. If the final force at joint 7 were also 1.00000 the resistance for joint 4, 3 bays from joint 7, would be only -8.75168. These values can also be checked from table 1. Since there is but one displacement of joint 4, that joint can have only one resistance. Hence both final forces cannot be 1.00000. Suppose, however, the final force at joint 7 were assumed to be $\phi = \dfrac{14.50990}{8.75168} = 1.65796$, and the resistances for joints 7, 6, 5, and 4 were computed as 1.65796 times the values given in table 1 for joints 0, 1, 2, and 3. Then these resistances combined with those already selected for joints 0, 1, 2, 3, and 4 would define a group displacement which would transfer an original force (as yet unknown) from joint 4 to the ends of the sequence. The magnitude of this original force can be computed from the relation

$$CL + R_n + CO1_n + \phi\ CO1_m = 0 \tag{1}$$

which represents the conditions for equilibrium at joint n. In this case, by using numerical values from table 1

$$CL - 14.50990 + 3.93826 + 1.65796 \times 2.22222 = 0$$

whence

$$CL = 6.86507$$

Values of ϕ and CL and also of CL/ϕ for various combinations of m and n and several values of k are given in table B-2.* They were computed from the tables similar to table 1 from which the values of table B-1* were copied. If it were desired to check a value of CL from the data in table B-1* it could be done from the relation

$$CL + F_n - 0.5k\ F_{n-1} - 0.5k\ \phi\ F_{m-1} = 0 \tag{2}$$

which is simply another form for equation (1).

Transfer of Force from Free to Fixed End of Sequence

At times it is desirable to transfer a force from one end of a sequence to the other end, only the latter being assumed fixed. If the final force at the fixed end joint is assumed to be unity, the associated characteristic load and resistances can be easily obtained from table B-1.* In developing that table, it was assumed that the force CL was to be transformed into two unit loads, each at one end of the sequence. If the value of CL of table B-1* is halved and only one-half of the sequence is considered, the resistances of that table will be applicable to the new transfer pattern, provided that the resistance listed for the joint at which the original force is applied is also halved. The procedure is analogous to treating a cantilever as one-half of a symmetrically loaded, simply supported beam. Thus if the fixed end of the sequence is joint 0 and the original force is applied at joint 4, that part of figure 4 which pertains to joints 0, 1, 2, and 3 will be applicable. At joint 4, the value of CO2 would be zero since the sequence is assumed to stop at that joint. At joint 3 the value of CO2 would be $-kR_4$ instead of $-kR_4/2$ since joint 3 is the only one of the sequence to which a force would be carried over from joint 4. This change in conditions at joint 3 results in a value for R_4 only one-half as large as before, since the numerical value of CO2 for joint 3 would not be changed. Then at joint 4, since the numerical value of R_4 has been halved and that of CO2 has become zero, the numerical value of the characteristic load that can be held in equilibrium will also be halved.

Transfer of Force from Intermediate Joint to Single Fixed End Joint

If only one joint of a sequence is fixed and a force is to be transferred to that joint from a joint other than one at the end of the sequence, the determination of required resistances becomes more complex. Between the fixed joint and that of original load application the resistances can be found from table B-1* in the same manner as if the latter were at an end of the sequence. The resistance for the joint of original load application may also be taken directly from the same table. Division by 2 is unnecessary since displacement of that joint produces carry-over forces at two adjacent joints of the sequence. If the transfer factor were unity the resistance at each joint of the unloaded sector between the free end and the joint of load application would be the same as that at the latter joint, except that at the end joint the figure so determined would be halved. When, however, the transfer factor is not unity, allowance must be made for the effect of leakage.

Figure 8 is an overbalance diagram for the portion of a sequence between the free end and the joint of load application for the type of transfer under consideration. In this diagram it is assumed that the load is applied at joint 4 and is to be carried to a joint to the right of those shown. The actual value of R_4 can be determined by the number of bays from that joint to the fixed joint. At each of the joints from 0 to 3 the algebraic sum of CO1, CO2, and R must be zero, and there are four unknown resistances. Since there are four equations with four unknowns, the equations can be solved simultaneously. The magnitude of the resistance for the joint at which the load is applied, however, depends on the number of bays from that joint to the fixed joint, and is independent of the number of bays in the unloaded sequence. Thus even though the number of joints of the unloaded sector between the loaded joint and the free end remained the same, it would be necessary to compute a separate set of factors for each number of bays between the loaded joint and the fixed end. It is more convenient to obtain a single set of values for each possible length of unloaded sector and to modify them according to the length of the loaded sector. This can be done as follows.

Assume the resistance for the free end joint is −1.00000. Then the resistances for the other joints can be computed serially in the same manner as those of table 1, as illustrated by table 26, which is developed for $k = 0.90$. Comparison of column (4) of table 26 and column (6) of table 1, in which are listed characteristic loads for the transfer of a centrally applied force to fixed end joints, will show that for any given joint number the numerical values are the same, though of opposite sign. An attempt to prove that this should always be true failed, but another investigation might be successful. Although no such proof was developed, the relation served as a useful check and made unnecessary a separate tabulation of values obtained from computations like those illustrated in table 26.

The resistances of column (4) of table 26 are relative and must be multiplied by the ratio ψ of the actual resistance for the loaded joint, as determined by its distance from the fixed joint, to the relative resistance for that joint as computed in table 26. The presence of the unloaded sector also has an effect on the size of the characteristic load, that is, the load which would produce a final force of unity at the fixed joint. If the loaded joint is n bays from the fixed joint and m bays from the free end joint, the carry-over forces at that joint will be $-k/2$ times R_{n-1} and $\psi k/2$ times CL_{m-1} from table B-1.* The characteristic load will therefore be the equilibrant of these carry-over forces and the resistance R_n from table B-1.* Table B-3* gives characteristic loads CL obtained in this manner and also the corresponding values of CL/ψ. Values of ψ are not given separately, since in practice resistances for the unloaded sector can be obtained most conveniently by multiplying the values of CL of table B-1* by the actual load W divided by CL/ψ from table B-3.*

Uniform Distribution of Single Force Applied at an End Joint

In many problems it is convenient to assume that the original load is a single force applied at one end of a sequence and the final forces are uniformly distributed. In this report the term "uniformly distributed" is used to signify that a load is uniformly distributed between bays and the portion associated with each bay divided equally between the joints between which it lies. Thus the joint loads at the intermediate bays are of the same size, but those at the end bays are only one-half as large as those at the intermediate joints. In order to define completely the transfer pattern, it is necessary to select the joint which is to be assumed fixed. The most convenient choice for computation of overbalance factors is the joint at the opposite end of the sequence from the original single force. The factors computed for the case when this joint is fixed can then be adjusted to allow for alternative choices for the fixed joint. A typical overbalance diagram for the transfer pattern just described is presented in figure 9.

For a given value of k the resistances for all except the end joint at which the original concentrated force is applied can be computed in the same manner as those for the first case considered. The first part of the computation table for $k = 0.90$ is shown in table 27 with operation notes to indicate the method of computation used. In this table, since the final force for joint 0 is 0.5 and CO1 and R_1 for that joint are both zero, $CO2_0$ must also be 0.5, and $R_1 = \frac{-0.5}{0.45} = -1.11111$. $R_0 = 0$ since joint 0 is, by hypothesis, not displaced. The fact that $CO1_0 = 0$ follows from the fact that there is no joint -1. Once these starting values have been inserted, the remaining values for columns (3), (4), and (5) can be computed by following the operations notes in the column headings.

The computations for columns (3) to (5) of table 27 are based on the relationship

$$R_i - 0.5kR_{i-1} - 0.5kR_{i+1} = 1 \tag{3}$$

This represents the equilibrium conditions for all joints except 0, 1, n–1, and n, where joint 0 is at the fixed end and joint n at the movable end of the sequence. The special conditions at joint 0 were taken into account in computing the starting values. Equation (3) with $i = 1$ can be used to compute R_2 since R_{i-1} is then R_0, which equals zero. When, however, $i = n-1$, equation (3) should be replaced by

$$R_{n-1} - 0.5kR_{n-2} - kR_n = 1 \tag{4}$$

and when $i = n$ it should be replaced by

$$CL + R_n - 0.5kR_{n-1} = 0.5 \tag{5}$$

Comparison of equations (3) and (4) will show that the numerical value of R_n obtained from equation (4) will be only one-half as great as that of R_{i+1} from equation (3). In other words the resistance for the end joint is only one-half as great as it would be if it were an intermediate joint of a longer sequence. Once the resistance for the end joint has been found, equation (5) can be used to find the characteristic load for the transfer pattern. Column (7) of table 27 gives the characteristic loads found in this manner for the various lengths of sequence. Resistances and characteristic loads found in this manner for various sequence lengths and transfer factors are listed in table B–4.*

Although table 27 was constructed for the distribution of a concentrated force, it can also be used for the transfer of an original distributed loading to one end of a sequence, the other end being assumed not to move. All that is necessary is to change the signs of the resistances. This is allowable because, if the quantities in the OF– and FF–rows of figure 9 were interchanged, conditions at joint 0 would require that $-0.5kR_1$ should be –0.5 instead of 0.5. Then R_1 would be 1.11111 instead of –1.11111. Equation (3) would be modified by placing the +1 term on the left– instead of the right–hand side, and the positions of CL and 0.5 in equation (5) would be interchanged. Thus the net result of these changes would be to change the signs of all values of R without affecting their magnitudes.

The difference between the characteristic loads of column (7) of table 27 and the corresponding joint numbers of column (1) is a measure of the leakage of force from the sequence under consideration to other joints. Thus if the sequence has four bays, out of a total load of 6.89222 pounds applied at joint 4, only 4.00000 pounds would be uniformly distributed between the joints of the sequence and 2.89222 pounds would be transferred to other joints. If now all the joints of the sequence were moved through the <u>same</u> distance, an additional load would be uniformly distributed between the joints of the sequence, with equal and opposite forces being developed at joints of other sequences. If the net force to be added to each intermediate joint is 1 pound and that at each end joint, 1/2 pound, the additional resistance to be developed at each intermediate joint must be $1/(1-k)$ pounds. In the case under consideration, therefore, the resistance to be added to each intermediate joint will be

$$\Delta R = \frac{2.89222}{4} \times \frac{1}{1-0.90} = 7.23055 \text{ pounds}$$

The resistance increment for each end joint will be one-half that value.

In table 28 the second column shows the resistances for distributing 6.89222 pounds applied at joint 4 when joint 0 is fixed. The third column gives the increments associated with the "block displacement" required to eliminate the leakage, and column (4), the total resistances for distribution of the load without net leakage.

It is not correct to say that the use of the resistances of column (4) would result in the uniform distribution of the original load of 6.89222 pounds without any leakage. There would be leakage of positive forces associated with the displacements corresponding to the resistances of column (2) and leakage of negative forces associated with the displacements implied by the resistances of column (3). In total magnitude these positive and negative forces would be equal, so they would balance each other and there would be no net leakage. Some of the joints of adjacent sequences, however, would be subjected to net positive and others to net negative leakage forces. Even so, it is often of advantage to use group displacements which would distribute a force without net leakage. The factors necessary for the purpose are also useful in computing the factors for related transfer patterns. These factors have been computed for various combinations of sequence length and transfer factor by the method used for constructing table 28 and are given in table B-7.*

It is often advisable to "verify" a set of resistances to make sure that they will produce the desired transfer pattern, before using them in practice. This has been done in columns (5) and (6) of table 28 to check the resistances of column (4). In general the final force at any joint is equal to the algebraic sum of the original force, the resistance for the joint, and the carry-over forces at the joint. The sum of the carry-over forces is normally $-k/2$ times the sum of the resistances for the two adjacent joints. If, however, one of the adjacent joints is at the end of the sequence, its resistance should be multiplied by $-k$ instead of $-k/2$, or twice its resistance should be multiplied by $-k/2$. In table 28 each value in column (5) is -0.45 times the sum of the adjacent values of column (4), except that the values for joints 0 and 4 are doubled. Thus

$$-0.45(6.11944 + 0) = -2.75375$$

$$-0.45(2 \times 3.61527 + 2.53920) = -4.39638$$

$$-0.45(6.11944 - 4.30577) = -0.81615$$

$$-0.45(2.53920 - 2 \times 7.96828) = 6.02881$$

$$-0.45(-4.30577 + 0) = 1.93760$$

The values of column (6) are the algebraic sums of those in columns (4) and (5) and represent the net forces that would be imposed on the constraints by the displacements corresponding to the resistances of column (4). At each of the intermediate joints this force is about 1.72305, almost exactly equal to the characteristic load of 6.89222 divided by four, the number of bays. If more significant figures had been used in the computations, these intermediate net joint loads would all have been of the same magnitude to six figures, and that magnitude would have been the characteristic load divided by the number of bays. The net load at joint 0 is one-half that at each intermediate joint. That at joint 4 may be considered as the resultant of a negative 6.892222, equilibrated by the positive original force of that magnitude and a positive load of $\frac{1.72305}{2} = 0.86152$ acting on the constraints. The validity of the factors of column (4) for distributing a force of 6.89222 pounds at joint 4 without net leakage is thus proved by the values of column (6).

Since $(1 - k)$ times each resistance developed represents leakage, if the net leakage is to be zero, the algebraic sum of the resistances of column (4) should be zero. In table 28 this sum is −0.00014, and that value reflects the accumulation of small numerical errors. In this case, however, it is too small to be of interest and no attempt was made to get more precise results.

In order to use resistances like those of column (4) of table 28, it is necessary to know the magnitude of the original force at the end joint or that of each final force at an intermediate joint. Both of these quantities are therefore given with the associated overbalance factors in table B-7.* The final joint loads could be computed in each case by dividing the characteristic load by the number of bays, but it is convenient to have both values given.

Often it is desirable to transform a single force imposed at an end joint into a uniformly distributed loading or, vice versa, with a joint assumed fixed which is not at the other end of the sequence. The cases of most interest are those where the fixed joint is the end joint at which the single force is applied and that where the fixed joint is either at the center of the sequence or 1/2 panel from the center. These cases can also be used by changing the signs of the resistances to transfer a uniformly distributed load to a joint which remains fixed. The resistances desired can easily be obtained by combining those from column (4) of table 28 (or its equivalent) for the joint to be assumed fixed and each of the resistances given in that column. The method is illustrated by table 29 in which resistances are computed for the cases in which joints 2 and 4 are assumed fixed. Computations for verifying the work are included in the table.

In this table the values of column (2) are taken from table 28, column (4). The third column gives the resistances associated with distribution of the load of 6.89222 pounds applied at joint 4 with joint 2 assumed fixed. Since the original resistance for joint 2 is 2.53920, that value is subtracted from the factors of column (2) for the intermediate joints 1 to 3 and one-half that value from the resistances for the end joints 0 and 4. The results are shown in column (3). Columns (4) and (5) are the verification of column (3). The validity of the resistances of column (3) is substantiated by the following properties of net loads in column (5). The net load at joint 0, the unloaded end joint, is one-half that at each intermediate joint. The net load at joint 4, the loaded end joint, is equal to minus the original characteristic load plus the net at joint 0. The net at each intermediate joint is equal to the characteristic load divided by the number of bays minus $(1 - k)$ times the quantity subtracted from the intermediate joint resistances of column (2). The necessity of satisfying these relations is a useful basis for checking the work. It is therefore convenient to make a record below the main table of the characteristic load CL computed by adding the net load for joint 0 to minus the net load for joint 4 and the joint load increment ΔJL equal to $(k - 1)$ times the value of R_m for the joint assumed fixed. The quantity ΔJL is the leakage per bay associated with the transfer under consideration. Leakage is considered positive when the sum of the final forces is <u>less</u> than that of the original forces.

The resistance for joint 4 in column (2) is −7.96828 and that value is subtracted from the resistance for the end joints 0 and 4 and twice that amount from the resistances for the intermediate joints 1 to 3. The results listed in column (6) are the resistances to be used if joint 4 is assumed fixed. These values are verified in columns (7) and (8). The values of CL and ΔJL shown in column (8) are obtained in the same manner as those for column (5), except that since it is an end joint that is assumed fixed, ΔJL is $(k - 1)$ times <u>twice</u> the value of R_m for joint 4. It may be noticed that this value of ΔJL is negative and the intermediate joint net loads exceed the characteristic load divided by the number of bays. In other words the leakage is negative and the sum of the final forces exceeds the original force. This is reasonable. If the single load is assumed to be the original load, the problem is to liquidate it by moving the joints 0 to 3 toward the fixed joint 4. Although some of the resistance to this movement will be transferred to joint 4 of the sequence, a large part will go to joints of adjacent sequences instead. Perhaps the situation can be seen more clearly if the distributed load is considered the original load, to be transferred to joint 4. Then it should easily be seen that a large part will leak out to the parallel sequences instead of going directly to joint 4.

Resistances like those of columns (3) and (6) of table 29 are given in table B-8.* In that table both the concentrated load at the end joint and the fraction of the distributed load acting at each intermediate joint are given. The fraction of the distributed load acting at each end joint is one-half that at an intermediate joint. Both of these loads need to be tabulated since both are needed to indicate the magnitude of the leakage effect. In using the table it is to be remembered that a "half-portion" of distributed load is assumed to act at the point of application of the concentrated load. In the verification therefore the net load at that joint will be minus the concentrated load plus the fraction of the distributed load.

In table B-8* the only intermediate joints assumed fixed are those at the center of the sequences of even numbers of bays and at the ends of the center bays of sequences of odd numbers of bays. If it is desired to assume some other intermediate bay fixed, compute the appropriate resistances and characteristic load from the data of table B-7* in the manner outlined in the foregoing paragraphs.

Transfer of Uniformly Distributed Load to Two End Joints

At times it is desirable to transfer a uniformly distributed load to the end joints of a sequence without relative displacement of those end joints. The resistances for such an operation would be the same in magnitude as those for the uniform distribution of equal forces imposed at the end joints but opposite in sign. The latter viewpoint is the more convenient for computing the numerical values for the resistances.

Two cases are of special interest. In practice the end joints will normally be fixed, but for purposes of computation it is better to start with the case in which the whole sequence is so displaced that there will be no net leakage. Assume that a force W imposed at joint n is to be uniformly distributed without net leakage. The necessary resistances can be determined by the methods outlined in the preceding section. Thus if $n = 4$ and $k = 0.9$ they can be taken directly from column (2) of table 29. The additional resistances needed to obtain uniform distribution of a force W imposed at joint 0 can also be obtained from the same column by reversing the order of enumeration of the joints. Thus the added resistance for any joint i would be the value given for joint $n - i$. In other words, the desired resistance for any joint i would be the sum of those given for joints i and $n - i$. For the case used for illustration these values are given in column (2) of table 30. They are verified in columns (3) and (4) of the same table. It will be noted that the value given for the characteristic load is the same as in table 29; that is, the characteristic load for this case is taken as the single force to be applied at each end of the sequence. The net load at each intermediate joint is equal to the characteristic load divided by one-half the number of bays and therefore twice the net load for an intermediate joint in column (6) of table 28. The characteristic load is also equal to one-half the net load at each intermediate joint minus (algebraically) the net load at either end joint. These relations are convenient for checking.

It will be noticed that the resistances of column (2) of table 30 are symmetrical about the midpoint of the sequence, and those for the end joints are equal. If all joints are displaced the amount required to make the end-joint resistances zero, the change in resistance at each intermediate joint will be minus twice the original resistances at the end joints. Resistances computed in this manner are given in column (5) of table 30 and verified in columns (6) and (7). Here again one-half an intermediate-joint net force minus an end-joint net is equal to the characteristic load of 6.89221. Each intermediate-joint net force is equal to the sum of the intermediate-joint net of 1.0000 assumed in constructing table 28 and 3.31670 found in the verification of column (6) of table 29. This was to be expected and means that, if a uniformly distributed load of 4.31670 per bay is transferred to the end joints without movement of those joints, 1.00000 per bay produces a force of 6.89221 at end joint 0, and the remaining 3.31670 per bay produces the force of 6.89221 at joint 4. The reason for this unequal division has been commented on in the justification for "negative leakage."

The actual leakage per bay in the force transfer under consideration is $3.44610 - 4.31670 = -0.87060$. It should also be equal to $(1 - k)$ times change in resistance at each intermediate joint in going from the transfer pattern without net leakage to that with the end joints fixed in position. It satisfies this criterion, since $(1 - 0.90) \times 2 \times 4.35301 = 0.87060$.

Characteristic loads, intermediate-joint loads, and resistances for the uniform distribution of equal forces applied at the end joints are given in tables B-5* and B-6.* Those in table B-5* are for the transfer without net leakage, all joints being assumed displaced but with zero relative displacement of the end joints. Those in table B-6* are for the case when there is no displacement of either end joint.

Uniform Distribution of Single Force Applied at an Intermediate Joint

In the development of tables 27 to 30 it was assumed that the characteristic load was applied at an end joint of the sequence, though the joint, if any, that remained undisplaced might vary. It is also desirable to have resistances for the uniform distribution of a single force that is applied at any joint of a sequence. The investigations made so far indicate that the most useful of such resistances would be those associated with such distribution without net leakage, and they can be obtained by superposition of transfer patterns already considered.

Consider for example that $n = 4$ and $k = 0.90$ and that it is desired to find the factors for uniform distribution of a single force imposed at joint 2. The computations for this case are shown in table 31. In column (2) are listed the resistances for column (4) of table 28 for the uniform distribution, without net leakage, of a load of 6.89222 pounds imposed at joint 4. This characteristic load and the resulting intermediate-joint load are listed in the same column for convenient reference. These factors apply to a transfer pattern that differs from that desired in that it includes a force of 6.89222 at joint 4 which is not desired and lacks the desired concentrated original force at joint 2. This can be corrected by the transfer of the original force from joint 4 to joint 2 by means of the factors of table B-1* and the procedure already outlined for the transfer of a force from the free to the fixed end of a sequence.

In this case the concentrated load is to be moved 2 bays from joint 4 to joint 2. The resistances of table 1 for a two-bay movement of this type are appropriate to a characteristic load of 2.93826/2 pounds, while the force it is desired to move amounts to 6.89222 pounds. The resistances of table 1 must therefore be multiplied by the ratio $\gamma = 6.98222 \times 2/2.93826 = 4.69136$. The resistances thus computed are given in column (3) of table 31. That at joint 2 will be zero since that joint will not have to be displaced. At joint 3, $4.69136 \times 2.22222 = 10.42523$ must be added, and at joint 4, $4.69136 \times 4.93826/2 = 11.58538$ must be added. The signs of these resistances are positive since the load to be moved is not the actual characteristic load at joint 4 but its opposite. The situation may be considered in this manner. By the joint movements represented by the resistances of column (2) a set of uniformly distributed positive forces on the constraints and a single negative force at joint 4 were developed. Since the presence of an original positive force at joint 4 was assumed, this negative force was equilibrated by the original force instead of acting against the constraints. Now, however, assume that the original positive force at joint 4 does not exist, so the negative force must act on the constraints at joint 4, and it is this negative force which is to be transferred to joint 2. By applying the positive resistances of column (3) at joints 3 and 4 the negative force at joint 4 is replaced by a somewhat smaller negative force at joint 2 which would be available to equilibrate an original positive force at that joint.

If the resistances of columns (2) and (3) were added, the resulting values would be appropriate for the uniform distribution of a single force applied at joint 2, but there would be net leakage although no joint would be fixed. It is desirable therefore to assume an additional block displacement to counteract the leakage accompanying the movement of the concentrated force. In order to do this, determine the algebraic sum of the resistances of column (3) (in this case, 22.00881), reverse the sign, and divide by the number of bays (in this case, 4) to get the change in resistance for each intermediate joint (in this case, -5.50220). The change for each end joint is one-half that for an intermediate joint. These changes are given in column (4) of table 31 and the algebraic sums of the values of columns (2), (3), and (4) are the desired resistances. They are verified in columns (6) and (7) of the table. From column (7) it will be seen that the size of the characteristic load has been changed (in this case, from 6.89222 to 4.69137 pounds). It will also be noted that the net load at joint 2, where the concentrated load was applied, is minus (n – 1) times the net load at the other intermediate joints and that the net load at each end joint is one-half that at the other intermediate joints. Numerical values of resistances with associated characteristic loads and normal intermediate-joint loads obtained in this manner are given in table B-7.*

If it should be desired to compute resistances for the uniform distribution of a single force imposed at an intermediate joint with a single selected joint assumed fixed, that can be done in essentially the same manner as that employed for computing the values in table B-8* for single forces imposed at end joints.

Liquidation of Symmetrical Uniformly Varying Loading

In some types of problem it will be found that the forces to be liquidated are directly proportional to their distances from the center of the sequence, except that those at the end joints of the sequence are only one-half as large as those which would satisfy the criterion of proportionality, and also the forces on one side of the center are of opposite sign to those on the other. In this report such a loading is termed a "symmetrical uniformly varying load." The algebraic sum of the forces constituting such a loading is zero, and the forces on one side of the center are sufficient to liquidate those on the other side, though there will be leakage if the transfer factor is other than unity. In computing resistances for the liquidation of such a loading, assuming no movement of the center of the sequence, it is desirable to consider two groups of cases, those in which there is an even number of bays and the original load at the center joint is zero and those in which there is an odd number of bays and the forces on the joints at the ends of the central bay are equal and opposite. It is also convenient to modify the system of numbering the joints, by making the number assigned to each joint equal to the number of bays to the joint subjected to an equal and opposite load and giving the joint number the same sign as the force applied at it. This makes the number of a joint twice the number of bays to the center of the sequence. It is further convenient to assume the magnitude of the load at each joint as equal to the number of the joint, except that the magnitude of each end-joint force is only one-half the joint number.

A general overbalance diagram for an even number of bays is shown in figure 10 and that for an odd number of bays in figure 11. Because of the symmetry of the loading it is evident that the value of R_0 in figure 10 must be zero and that every value of R_i will be equal to $-R_{-i}$. From both figures it can be seen that for any value of i except 0, 1, 2, 3, or n the following relation holds, based on conditions at joint $i - 2$.

$$\frac{k}{2} R_i = R_{i-2} - \frac{k}{2} R_{i-4} + i - 2 \qquad (6)$$

whence

$$R_i = \frac{2}{k}(R_{i-2} + i - 2) - R_{i-4} \tag{7}$$

Also from conditions at joint $n - 2$,

$$R_n = \frac{1}{k}(R_{n-2} + n - 2) - \frac{1}{2}R_{n-4} \tag{8}$$

Thus R_n is just one-half the value it would be if joint n, instead of being an end joint, were an intermediate joint of a longer sequence. From conditions at joint n,

$$R_n - \frac{k}{2} R_{n-2} + \frac{n}{2} = 0 \tag{9}$$

But $R_n = 0.5 R_i$ for $i = n$, so, for $i = n$,

$$R_{i=n} - k R_{i-2} + i = 0 \tag{10}$$

It is not possible to solve the equations indicated by the joint conditions of figures 10 and 11 serially. The practical procedure is to find each value of R_i in terms of R_1 or R_2 by successive use of equation (7) and to use equation (10) to obtain the values of R_1 or R_2 corresponding to each value of n. Then the values of each R_i for each value of n can be computed.

When n is even, $R_0 = 0$ by symmetry, and R_4 can be computed in terms of R_2 from conditions at joint 2 by means of equation (7). When n is odd, the equation representing conditions at joint 1 is

$$R_3 = \left(1 + \frac{2}{k}\right)R_1 + \frac{2}{k} \tag{11}$$

and this must be used to start the computations for R_i where i is odd. The method used for computing the resistances in terms of R_2 or R_1 is illustrated by tables 32 and 33 for which it is assumed that $k = 0.90$. In these tables the quantities for each joint are placed in a separate column instead of a separate row, but that is done solely because, with the small number of joints considered in each table, space was saved by so doing.

Tables 34 and 35 indicate the method of computing the actual resistances after the value of R_2 or R_1 has been determined from table 32 or 33. In table 34 it is assumed that $n = 8$. In columns (2) and (3) are given the values of R_i from table 32. In row 11 of table 32 it was found that for $n = 8$, R_2 would be -13.97000. The actual resistances for joints 2, 4, and 6 are therefore found by subtracting

13.97000 times the value in column (2) from the value in column (3). The results are given in column (5). The same procedure was followed in the 8_i-row of table 34, but joint 8 is an end joint and its resistance is only one-half that found by the foregoing procedure. The correct value for that joint is therefore given in the 8_n-row. The last two columns of that table are a verification of the factors of column (5). In this verification the resistance on the 8_i-row is ignored, that on the 8_n-row being used. In finding the value of ΣCO for row 6, however, since joint 8 is an end joint, its resistance must be multiplied by $-k$ instead of $-k/2$, so the simplest practical procedure is to multiply the sum of the values from rows 4 and 8_i by -0.45 to get 30.25206. It is to be noted that, except for insignificant errors in the computations, the net effect of the displacements producing the resistances of column (5) is to develop forces of $-i$ pounds at each joint except joint n where the force is $-n/2$ pounds. Therefore if these displacements were imposed on a sequence subjected to the type of loading under consideration, that loading would be liquidated, except for the leakage. In such a group displacement the joints subjected to negative forces would be moved symmetrically to those with positive original forces. As a result the forces carried over to joint 0 from joints 2 and -2 would neutralize each other.

If there were only two bays and three joints in the sequence, n would be equal to 2. The loads at the end joints would be only 1 and -1, however, since the end joints are assumed to have loads equal to but one-half the joint number. From table 32 the paradoxical situation is seen that the value of R_2 is given as -2 though it is obvious that a displacement sufficient to develop a resistance of -1 should be sufficient to liquidate a force of only 1 at that joint. The values of R_2 given in table 32, however, are for use in computing values of R_i where i is less than n. When $n = 2$, this value of R_2 must be halved, and thus the correct value of R_2 to be used for actual liquidation is -1.00000.

Resistances for the liquidation of the type of loading considered in this section are given in table B-9.* The characteristic loads shown in this table are the arithmetic sums of the original forces of the same sign in the loadings considered.

Correction of Resistance Tables

Most of the other resistances given in tables B-1* to B-9* are obtained by manipulation of the resistances of tables B-1* and B-4.* Any errors in computing the resistances for these two tables therefore have widespread results. Because of the method of computation used for these basic tables, as the value of i or n increases the associated values of resistance are more and more likely to be erroneous. Thus in table B-4*, for $k = 0.90$ and $i = 30$ the resistance R_i has seven figures before the decimal point and the characteristic load CL has six.

In order to obtain these quantities accurately to five figures after the decimal point by the method outlined, it would be necessary to use 12 significant figures throughout the computations. That would have been entirely impracticable.

The actual procedure followed was to obtain approximate values by the method already outlined and to correct these trial values by an additional computation. By this device the computation of the desired factors for large values of i and n and relatively small values of k to the desired precision was made practicable. The method used for correcting the trial values is described as follows.

Table 36 is a record of computations for correcting the trial values of R_i for $k = 0.88$ in table B-4.* The original or trial values of R_i in column (2) were computed in the manner illustrated by table 27. The first step is to verify these resistances. The values of ΣCO in column (3) were computed in the usual manner. In column (4), instead of the algebraic sum of the values of columns (2) and (3), the difference between that net sum and the final force required by the transfer pattern under investigation is recorded. In this case the transfer pattern requires the value of 0.50000 at joint 0 and 1.00000 at each of the other joints so the value for column (4) is readily determined. Thus column (4) shows that if the resistances of column (2) were used the final forces would not be uniformly distributed, but there would be an excess of 0.00001 at joint 2, 0.00001 at joint 3, 0.00003 at joint 4, and so forth.

The resistances required to eliminate these errors can be computed by considering them as a new set of original forces that is to be transferred to the free end of the sequence. At joint 2 the error is 1; therefore the value of $\Delta CO1$ must be -1 at this joint and that of $\Delta CO2$ at joint 4 must also be -1. Since, for this table, $k = 0.88$, the value of ΔR_i for joint 3 must be $1/0.44 = 2$. At joint 3, the value of $\Delta CO2$ is zero since the error at joint 1 and therefore the value of $\Delta CO1$ for that joint are both zero. The error for joint 3 is 1. Then the value of $\Delta CO1$ for joint 3 will be $-(1 + 2) = -3$ and the value of ΔR_i for joint 4 will be $3/0.44 = 7$. The other values in columns (5), (6), and (7) were obtained in the same manner with the algebraic sum of the values in any row for columns (4) to (7) equal to zero. The corrected values of R_i in column (8) were obtained by adding the values in columns (2) and (7) after multiplying the values in column (7) by 10^5. These values of column (8) are verified in columns (9) and (10). Again the errors are shown in the second of the verification columns instead of the actual net forces. In this case the maximum error given in column (10) is only 0.00001, which is negligible.

Table 36 includes only the first 11 rows of a complete correction table and some of the values for joint 10 are based on values for joint 11 that are not shown. Since the maximum error given in column (10) is negligible, no further correction is needed. In some of the tables, particularly where values were computed for large values of i, the errors of column (10) were too large. This situation was taken care of by repeating the process until the errors were negligible. After the values of the resistances had been corrected in this manner, corrected values of the characteristic loads were computed by the application of equation (5).

The method described was used to check all the values given in table B-4.* The same method could have been used to check those of table B-1*, but it had not been developed until after the values in table B-1* had been checked by the construction of tables similar to table 26. Some of the values of table B-1*, however, were later checked by this system.

REFERENCES

1. Hoff, N. J., Levy, Robert S., and Kempner, Joseph: Numerical Procedures for the Calculation of the Stresses in Monocoques. I – Diffusion of Tensile Stringer Loads in Reinforced Panels. NACA TN No. 934, 1944.

2. Hoff, N. J., and Kempner, Joseph: Numerical Procedures for the Calculation of the Stresses in Monocoques. II – Diffusion of Tensile Stringer Loads in Reinforced Flat Panels with Cut-Outs. NACA TN No. 950, 1944.

Table 1

COMPUTATION OF OVERBALANCE FACTORS FOR TRANSFER OF CONCENTRATED LOAD AT CENTER JOINT TO FIXED END JOINTS

$[k = 0.90]$

(1) Joint	(2) FL Assumed	(3) R $-2(5)_{i-1}/0.9$	(4) CO1 $(5)_{i-2}$	(5) CO2 $(2)_i-(3)_i+(4)_i$	(6) CL $(5)_i-(4)_i$
0	1.00000	0	0	1.00000	0
1	0	-2.22222	0	2.22222	2.22222
2	0	-4.93826	1.00000	3.93826	2.93826
3	0	-8.75168	2.22222	6.52946	4.30724
4	0	-14.50990	3.93826	10.57164	6.63338

Table 2

LIQUIDATION TABLE FOR ANALYSIS OF FLAT PANEL

Row	Step	Displacements	Y_A	Y_E	Y_J	Y_N	Y_B	Y_F	Y_K	Y_O
1	0	Original forces	120.0			-60.0				-60.0
2	1	$v_A = v_E = v_J = v_N = 2.500$	-10.0	-20.0	-20.0	-10.0	10.0	20.0	20.0	10.0
3			110.0	-20.0	-20.0	-70.0	10.0	20.0	20.0	-50.0
4	2	$v_A = 2.760$	-140.4	129.2			5.5	5.5		
5		$v_E = 0.651$	30.4	-66.1	30.4		1.3	2.6	1.3	
6		$v_J = -0.921$		-43.1	93.5	-43.1		-1.8	-3.6	-1.8
7		$v_N = -2.222$			-104.0	113.0			-4.5	-4.5
8			0	0	-0.1	-0.1	16.8	26.3	13.2	-56.3
9	3	$v_B = 0.740$	1.5	1.5			-40.9	37.9		
10		$v_F = 0.473$	0.9	1.9	1.0		24.2	-52.2	24.2	
11		$v_K = -0.228$		-0.4	-0.9	-0.5		-11.7	25.2	-11.7
12		$v_O = -1.230$			-2.5	-2.5			-63.0	67.9
13			2.4	3.0	-2.5	-3.1	0.1	0.3	-0.4	-0.1
14	4	$v_A = 0.096$	-4.9	4.5			0.2	0.2		
15		$v_E = 0.052$	2.5	-5.3	2.5		0.1	0.2	0.1	
16		$v_J = -0.048$		-2.2	5.0	-2.3		-0.1	-0.2	-0.1
17		$v_N = -0.104$			-4.9	5.3			-0.2	-0.2
18			0	0	0.1	-0.1	0.4	0.6	-0.7	-0.4

Table 3

DESIGN OF GROUP DISPLACEMENT FOR STEP 2

(1) Joint	(2) OF	(3) CL	(4) AR	(5) CR	(6) A	(7) E	(8) J	(9) N
A	120.0	3.888	30.86	-3.625	-111.9			
E				-.837		-25.8		
J				2.613			80.6	
N				1.850				57.1
A				1.850	-28.5			
E				2.613		-40.3		
J				-.837			12.9	
N	-60.0	3.888	-15.43	-3.625				55.9
				R	-140.4	-66.1	93.5	113.0
				R/v	-50.8	-101.6	-101.6	-50.8
				v	2.760	.651	-.921	-2.222

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Table 4

DESIGN OF GROUP DISPLACEMENT FOR STEP 3

(1) Joint	(2) OF	(3) CL	(4) AR	(5) CR	(6) A	(7) E	(8) J	(9) N
B	16.8	3.888	4.32	-3.625	-15.7			
F				-.837		-3.6		
K				2.613			11.3	
O				1.850				8.0
B				-.385	-2.8			
F	26.3	3.577	7.35	-2.133		-15.7		
K				1.317			9.7	
O				1.202				8.8
B				1.202	4.4			
F				1.317		4.9		
K	13.2	3.577	3.70	-2.133			-7.9	
O				-.385				-1.4
B				1.850	-26.8			
F				2.613		-37.8		
K				-.837			12.1	
O	-56.3	3.888	-14.49	-3.625				52.5
				R	-40.9	-52.2	25.2	67.9
				R/v	-55.2	-110.4	-110.4	-55.2
				v	.740	.473	.228	-1.230

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Table 5

DESIGN OF GROUP DISPLACEMENT FOR STEP 4

(1) Joint	(2) OF	(3) CL	(4) AR	(5) CR	(6) A	(7) E	(8) J	(9) N
A	2.4	3.89	0.618	−3.625	−2.24			
E				−.837		−0.52		
J				2.613			1.61	
N				1.850				1.14
A				−.385	−.32			
E	3.0	3.58	.838	−2.133		−1.79		
J				1.317			1.10	
N				1.202				1.01
A				1.202	−.84			
E				1.317		−.92		
J	−2.5	3.58	−.700	−2.133			1.49	
N				−.385				.26
A				1.850	−1.47			
E				2.613		−2.08		
J				−.837			.67	
N	−3.1	3.89	−.798	−3.625				2.89
				R	−4.87	−5.31	4.87	5.30
				R/v	−50.8	−101.6	−101.6	−50.8
				v	.096	.052	−.048	−.104

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Table 6

COMPARISON OF COMPUTED DISPLACEMENTS

Joint	Table 2	Hoff	Modified	Difference
A	5.356	4.91	4.883	−0.027
B	.740	.27	.267	−.003
E	3.203	2.75	2.730	−.020
F	.473	0	0	0
J	1.531	1.06	1.058	−.002
K	−.228	−.72	−.701	.019
N	.174	−.33	−.299	.031
O	−1.230	−1.74	−1.703	.037

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Table 7

OVERBALANCE TABLE FOR TRANSFER OF CONCENTRATED FORCE TO FIXED END JOINTS WITH ORIGINAL FORCE AT CENTER JOINT

[$n = 6$; CL 1; displacement curve has two straight segments; $R_i = -i$ for $i \leq n$; values for $n < i < 2n$ not listed since they are symmetrical to those shown]

Joint	OF	R	CO1	CO2	FF
0	--	0	0	0.5	0.5
1	--	-1	0	1.0	---
2	--	-2	0.5	1.5	---
3	--	-3	1.0	2.0	---
4	--	-4	1.5	2.5	---
5	--	-5	2.0	3.0	---
6	--	-6	2.5	2.5	---

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Table 8

OVERBALANCE TABLE FOR TRANSFER OF CONCENTRATED

FORCE TO FIXED END JOINTS WITH ORIGINAL

FORCE AT CENTER JOINT

[$n = 5$; $m = 3$; $CL = m + n$; displacement curve has two straight segments; $R_i = -2im$ for $0 < i < n$; $R_i = -2(m + n - i)n$ for $n < i < m + n$]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	3	3
1	--	-6	0	6	--
2	--	-12	3	9	--
3	--	-18	6	12	--
4	--	-24	9	15	--
5	8	-30	12	10	--
6	--	-20	15	5	--
7	--	-10	10	0	--
8 F	--	0	5	0	5

Table 9

OVERBALANCE TABLE FOR UNIFORM DISTRIBUTION OF CONCENTRATED

FORCE AT CENTER JOINT WITH END JOINTS FIXED

[n = 5; CL = 2n; displacement curve has two parabolic segments with apexes at joints 0 and 2n; $R_i = -i^2$ where $0 < i < n$]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	0.5	0.5
1	--	-1	0	2.0	1.0
2	--	-4	0.5	4.5	1.0
3	--	-9	2.0	8.0	1.0
4	--	-16	4.5	12.5	1.0
5	10	-25	8.0	8.0	1.0
6	--	-16	12.5	4.5	1.0
7	--	-9	8.0	2.0	1.0
8	--	-4	4.5	.5	1.0
9	--	-1	2.0	0	1.0
10 F	--	0	.5	0	.5

Table 10

OVERBALANCE TABLE FOR TRANSFER OF UNIFORMLY

DISTRIBUTED LOAD TO TWO FIXED END JOINTS

[$n = 8$; CL = n; displacement curve is parabolic with apex at joint $n/2$; $R_i = -i\ (n - i)$]

Joint	OF	R	CO1	CO2	FF
0 F	0.5	0	0	3.5	4.0
1	1.0	-7	0	6.0	---
2	1.0	-12	3.5	7.5	---
3	1.0	-15	6.0	8.0	---
4	1.0	-16	7.5	7.5	---
5	1.0	-15	8.0	6.0	---
6	1.0	-12	7.5	3.5	---
7	1.0	-7	6.0	0	---
8 F	.5	0	3.5	0	4.0

Table 11

OVERBALANCE TABLE FOR TRANSFER OF TWO EQUAL SYMMETRICALLY

LOCATED FORCES TO TWO FIXED END JOINTS

[m = 3 (distance between forces); n = 4 (distance from force to nearest end); CL = 1 (each original force); displacement curve composed of three straight segments; $R_i = -2i$ when $0 < i < n$; $R_i = -2n$ when $n < i < m + n$]

Joint	OF	R	CO1	CO2	FF
0 F	---	0	0	1	1
1	---	-2	0	2	--
2	---	-4	1	3	--
3	---	-6	2	4	--
4	1.0	-8	3	4	--
5	---	-8	4	4	--
6	---	-8	4	4	--
7	1.0	-8	4	3	--
8	---	-6	4	2	--
9	---	-4	3	1	--
10	---	-2	2	0	--
11 F	---	0	1	0	1

Table 12

OVERBALANCE TABLE FOR UNIFORM DISTRIBUTION OF TWO EQUAL

SYMMETRICALLY LOCATED FORCES WITH

BOTH END JOINTS FIXED

[m = 5 (distance between forces); n = 4 (distance from force to nearest end); CL = n + m/2 (each original force); displacement curve has three parabolic segments with apexes at end joints and halfway between; $R_i = -i^2$ when $0 \leq i < n$;
$R_i = -n^2 + \frac{m^2}{4} - \left(\frac{m + 2n - 2i}{2}\right)^2$; when m is even
$F_{n+m/2}$ (at middle joint) $= -n^2 + m^2/4$]

Joint	OF	R	CO1	CO2	FF
0 F	---	0	0	0.5	0.5
1	---	-1	0	2.0	1.0
2	---	-4	0.5	4.5	1.0
3	---	-9	2.0	8.0	1.0
4	6.5	-16	4.5	6.0	1.0
5	---	-12	8.0	5.0	1.0
6	---	-10	6.0	5.0	1.0
7	---	-10	5.0	6.0	1.0
8	---	-12	5.0	8.0	1.0
[a]9	6.5	-16	6.0	4.5	1.0

[a]Values for joints 10 to 13 not shown on account of symmetry.

Table 13

OVERBALANCE TABLE FOR TRANSFER OF TWO EQUAL BUT

OPPOSITE FORCES TO TWO FIXED END JOINTS

[s = 9 (total length of sequence); m = 4 (distance between forces); n = 3 (distance from joint 0 to nearest force); CL = $\pm$ s; displacement curve has three straight segments; $R_i = -2im$ when $0 < i < n$; equal and opposite loads need not be equidistant from the middle of the sequence of joints]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	4	4
1	--	-8	0	8	--
2	--	-16	4	12	--
3	9	-24	8	7	--
4	--	-14	12	2	--
5	--	-4	7	-3	--
6	--	6	2	-8	--
7	-9	16	-3	-4	--
8	--	8	-8	0	--
9 F	--	0	-4	0	-4

Table 14

OVERBALANCE TABLE FOR TRANSFER OF EQUAL AND OPPOSITE UNIFORMLY

VARYING LOADS TO TWO FIXED END JOINTS

(a) n, even.

[n = 12; CL = 360; each value of OF = np where n is total number of panels and p is number of panels to equal but opposite OF; FF for joints 0 and n from table 15; displacement curve is cubic parabola]

Joint	OF	R	CO1	CO2	FF
0 F	0	0	0	220	220
1	120	-440	0	320	----
2	96	-640	220	324	----
3	72	-648	-320	256	----
4	48	-512	324	140	----
5	24	-280	256	0	----
6	0	0	140	-140	----
7	-24	280	0	-256	----
8	-48	512	-140	-324	----
9	-72	648	-256	-320	----
10	-96	640	-324	-220	----
11	-120	440	-320	0	----
12 F	0	0	-220	0	-220

Table 14 - Concluded

OVERBALANCE TABLE FOR TRANSFER OF EQUAL AND OPPOSITE UNIFORMLY VARYING LOADS TO TWO FIXED END JOINTS - Concluded

(b) n, odd.

[n = 11; CL = 275; each value of OF = np computed in same manner as for even values of n; FF for joints 0 and n from table 15]

Joint	OF	R (a)	CO1	CO2	FF (b)
0 F	0	0	0	165	165
1	99	-330	0	231	---
2	77	-462	165	220	---
3	55	-440	231	154	---
4	33	-308	220	55	---
5	11	-110	154	-55	---
6	-11	110	55	-154	---
7	-33	308	-55	-220	---
8	-55	440	-154	-231	---
9	-77	462	-220	-165	---
10	-99	330	-231	0	---
11 F	0	0	-165	0	165

[a]Formulas for resistances are too complex for practical use.

[b]It is better to use the final forces from table 15 and follow the procedure used in connection with figure 3.

Table 15

CHARACTERISTIC LOADS AND FINAL FORCES FOR OVERBALANCE TABLES FOR TRANSFER OF EQUAL AND OPPOSITE UNIFORMLY VARYING LOADS TO TWO FIXED END JOINTS

n	CL	$FF_{0,n}$	n	CL	$FF_{0,n}$	n	CL	$FF_{0,n}$
3	3	1	16	896	560	29	5,684	3,654
4	8	4	17	1088	680	30	6,300	4,060
5	20	10	18	1296	816	31	6,975	4,495
6	36	20	19	1539	969	32	7,680	4,960
7	63	35	20	1800	1040	33	8,448	5,456
8	96	56	21	2100	1330	34	9,248	5,984
9	144	84	22	2420	1540	35	10,115	6,545
10	200	120	23	2783	1771	36	11,016	7,140
11	275	165	24	3168	2024	37	11,988	7,770
12	360	220	25	3600	2300	38	12,996	8,436
13	468	286	26	4056	2600	39	14,079	9,139
14	588	364	27	4563	2925	40	15,200	9,880
15	735	455	28	5096	3276	41	16,400	10,583

Table 16

OVERBALANCE TABLE FOR TRANSFER OF SINGLE FORCE

FROM MOVABLE TO FIXED END

OF A SEQUENCE OF JOINTS

[n = 6; CL = 1; displacement curve is straight line; R_i = -2i, except R_n = -n]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	1	1
1	--	-2	0	2	--
2	--	-4	1	3	--
3	--	-6	2	4	--
4	--	-8	3	5	--
5	--	-10	4	6	--
6	1	-6	5	0	--

Table 17

OVERBALANCE TABLE FOR TRANSFER OF SINGLE FORCE FROM

INTERMEDIATE JOINT TO FIXED END OF

A SEQUENCE OF JOINTS

[$n = 5$; CL = 1; displacement curve is sloping line between original and final load positions and horizontal line between free end and original load position; $R_i = -2i$ when $i \leq n$; $R_i = -2n$ when $i \geq n$, except $R_i = -n$ at free end]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	1	1
1	--	-2	0	2	--
2	--	-4	1	3	--
3	--	-6	2	4	--
4	--	-8	3	5	--
5	1	-10	4	5	--
6	--	-10	5	5	--
7	--	-10	5	5	--
8	--	-10	5	5	--
9	--	-10	5	5	--
10	--	-10	5	5	--
11	--	-5	5	0	--

Table 18

OVERBALANCE TABLE FOR TRANSFER OF SINGLE FORCE

FROM INTERMEDIATE JOINT TO MOVABLE END OF

SEQUENCE WITH ONE FIXED END JOINT

[Resistances are the same as in table 16 except for change of sign]

Joint	OF	R	CO1	CO2	FF
-3 F	--	--	--	--	--
-2	--	--	--	--	--
-1	--	--	--	--	--
0	1	--	--	-1	--
1	--	2	--	-2	--
2	--	4	-1	-3	--
3	--	6	-2	-4	--
4	--	8	-3	-5	--
5	--	10	-4	-6	--
6	--	6	-5	0	1

Table 19

OVERBALANCE TABLE FOR UNIFORM DISTRIBUTION OVER

ENTIRE SEQUENCE WITH ONE FIXED END JOINT

OF SINGLE LOAD AT MOVABLE END JOINT

[n = 9; CL = 9; displacement curve is parabolic with apex at fixed joint; $R_i = -i^2$, except $R_n = n^2/2$ at movable end]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	0.5	0.5
1	--	-1	0	2.0	1.0
2	--	-4	0.5	4.5	1.0
3	--	-9	2.0	8.0	1.0
4	--	-16	4.5	12.5	1.0
5	--	-25	8.0	18.0	1.0
6	--	-36	12.5	24.5	1.0
7	--	-49	18.0	32.0	1.0
8	--	-64	24.5	40.5	1.0
9	9	-40.5	32.0	0	.5

Table 20

OVERBALANCE TABLE FOR UNIFORM DISTRIBUTION OVER ENTIRE SEQUENCE WITH ONE FIXED END JOINT OF SINGLE LOAD AT INTERMEDIATE JOINT

[$n = 6$; $m = 3$; $CL = m + n$; displacement curve has two parabolic segments with apexes at ends of sequence and common ordinate at position of original force; $R_i = -i^2$ when $i < n$; $R_i = -n^2 + m^2 - (m + n - i)^2$ when $i > n$, except $R_i = -\frac{1}{2}(n^2 - m^2)$ at $i = m + n$]

Joint	OF	R	CO1	CO2	FF
0 F	--	0	0	0.5	0.5
1	--	-1	0	2.0	1.0
2	--	-4	0.5	4.5	1.0
3	--	-9	2.0	8.0	1.0
4	--	-16	4.5	12.5	1.0
5	--	-25	8.0	18.0	1.0
6	9	-36	12.5	15.5	1.0
7	--	-31	18.0	14.0	1.0
8	--	-28	15.5	13.5	1.0
9	--	-13.5	14.0	0	.5

Table 21

OVERBALANCE TABLE FOR TRANSFER OF UNIFORMLY

DISTRIBUTED LOAD TO ONE FIXED END JOINT

[$n = 9$; $CL = n$; displacement curve is parabola with apex at movable end joint; $R_0 = -0.5n^2$; $R_i = -n^2 + i^2$]

Joint	OF	R	CO1	CO2	FF
0	0.5	-40.5	0	40	--
1	1.0	-80	40.5	38.5	--
2	1.0	-77	40.0	36.0	--
3	1.0	-72	38.5	32.5	--
4	1.0	-65	36.0	28.0	--
5	1.0	-56	32.5	22.5	--
6	1.0	-45	28.0	16.0	--
7	1.0	-32	22.5	8.5	--
8	1.0	-17	16.0	0	--
9 F	.5	0	8.5	0	9

Table 22

OVERBALANCE TABLE FOR TRANSFER OF UNIFORMLY

DISTRIBUTED LOAD TO MOVABLE END JOINT

WITH OTHER END JOINT FIXED

[$n = 8$; $CL = n$; displacement curve is parabola with apex at movable end joint; $R_i = i^2$, except $R_n = 0.5n^2$]

Joint	OF	R	CO1	CO2	FF
0 F	0.5	0	0	-0.5	--
1	1.0	1	0	-2.0	--
2	1.0	4	-0.5	-4.5	--
3	1.0	9	-2.0	-8.0	--
4	1.0	16	-4.5	-12.5	--
5	1.0	25	-8.0	-18.0	--
6	1.0	36	-12.5	-24.5	--
7	1.0	49	-18.0	-32.0	--
8	.5	32	-24.5	0	8

Table 23

OVERBALANCE TABLE FOR TRANSFER OF UNIFORMLY DISTRIBUTED

LOAD TO A MOVABLE INTERMEDIATE JOINT

WITH ONE END JOINT FIXED

[$m = 6$ (distance FF to movable end); $n = 4$ (distance FF to fixed end); $CL = m + n$; displacement curve has two parabolic segments with apexes at end joints; $R_i = i^2$ when $i \leq n$; $R_{m+n} = \frac{n^2 - m^2}{2}$ at movable end; $R_i = n^2 - m^2 + (n + m - i)^2$ when $n < i < m + n$]

Joint	OF	R	CO1	CO2	FF
0 F	0.5	0	0	-0.5	----
1	1.0	1	0	-2.0	----
2	1.0	4	-0.5	-4.5	----
3	1.0	9	-2.0	-8.0	----
4	1.0	16	-4.5	-2.5	10.0
5	1.0	5	-8.0	-2.0	----
6	1.0	-4	-2.5	-5.5	----
7	1.0	-11	2.0	8.0	----
8	1.0	-16	5.5	9.5	----
9	1.0	-19	8.0	10.0	----
10	.5	-10	9.5	0	----

Table 24

OVERBALANCE TABLE FOR TRANSFER OF LOAD UNIFORMLY

DISTRIBUTED OVER PART OF SEQUENCE ADJOINING

FIXED END JOINT TO THAT JOINT

[m = 3 (length of unloaded segment); n = 6 (length of loaded segment); CL = n; displacement curve is horizontal line for unloaded segment tangent to parabola for loaded segment; at movable end $R = -0.5n^2$; over remainder of unloaded segment $F = -n^2$; over loaded segment $F_i = -n^2 + i^2$]

Joint	OF	R	CO1	CO2	FF
-3	---	-18	0	18.0	--
-2	---	-36	18.0	18.0	--
-1	---	-36	18.0	18.0	--
0	0.5	-36	18.0	17.5	--
1	1.0	-35	18.0	16.0	--
2	1.0	-32	17.5	13.5	--
3	1.0	-27	16.0	10.0	--
4	1.0	-20	13.5	5.5	--
5	1.0	-11	10.0	0	--
6 F	.5	0	5.5	0	6

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Table 25

OVERBALANCE TABLE FOR MUTUAL LIQUIDATION OF TWO EQUAL

BUT OPPOSITE FORCES WITH ONE FIXED END JOINT

[$n = 4$ (distance between original loads); $CL = 1$; displacement curve is sloping line between original loads; $R_i = 2i$ when $i \leq n$; $R_i = 2n$ when $i \geq n$, except $\bar{R} = n$ at movable end]

Joint	OF	R	CO1	CO2	FF
-3	--	-	--	--	--
-2	--	-	--	--	--
-1	--	-	--	--	--
0	1	0	0	-1	--
1	--	2	0	-2	--
2	--	4	-1	-3	--
3	--	6	-2	-4	--
4	-1	8	-3	-4	--
5	--	8	-4	-4	--
6	--	4	-4	0	--

Table 26

RELATIVE OVERBALANCE FACTORS FOR UNLOADED SECTOR OF A SEQUENCE

[k = 0.90]

(1) Joint	(2) CO1	(3) CO2	(4) R
0	0	1.00000	-1.00000
1	0.90000	1.32222	-2.22222
2	1.00000	1.93826	-2.93826
3	1.32222	2.98502	-4.30724
4	1.93826	4.69511	-6.63337

Table 27

OVERBALANCE-FACTOR COMPUTATIONS FOR UNIFORM DISTRIBUTION OF CONCENTRATED FORCE AT FREE END OF A SEQUENCE WITH OTHER END ASSUMED FIXED

[k = 0.90]

(1) Joint	(2) FL Assumed	(3) CO2 $(4)_{i-2}$	(4) CO2 $1 - (3)_i - (5)_i$	(5) R_i $-2\ (4)_i/0.90$	(6) R_n $0.5\ (5)_i$	(7) CL $0.5 - (3)_i - (6)_i$
0	0.5	0	0.50000	0	0	0
1	1.0	0	2.11111	-1.11111	-0.55555	1.05555
2	1.0	0.50000	5.19135	-4.69135	-2.34568	2.34568
3	1.0	2.11111	10.42521	-11.53632	-5.76816	4.15705
4	1.0	5.19135	18.97578	-23.16711	-11.58356	6.89222
-	---	-------	--------	--------	--------	-------

Table 28

OVERBALANCE FACTORS FOR UNIFORM DISTRIBUTION WITHOUT NET LEAKAGE

OF A FORCE APPLIED AT THE END JOINT OF A SEQUENCE

[Sequence length, 4 bays; transfer factor, 0.9]

(1) Joint	(2) R_0	(3) ΔR	(4) R	(5) ΣCO	(6) Net
0	0	3.61527	3.61527	-2.75375	0.86152
1	-1.11111	7.23055	6.11944	-4.39638	1.72306
2	-4.69135	7.23055	2.53920	-.81615	1.72305
3	-11.53632	7.23055	-4.30577	6.02881	1.72304
4	-11.58356	3.61528	-7.96828	1.93760	-6.03068
		Sum	-0.00014		

Table 29

OVERBALANCE FACTORS FOR UNIFORM DISTRIBUTION OF A SINGLE FORCE AT

AN END JOINT WITH VARIOUS JOINTS ASSUMED FIXED

[$n = 4$, $k = 0.90$]

(1) Joint	(2) R_m	(3) R_2	(4) ΣCO	(5) Net	(6) R_4	(7) ΣCO	(8) Net
0	3.61527	2.34567	-1.61111	0.73456	11.58355	-9.92520	1.65835
1	6.11944	3.58024	-2.11110	1.46914	22.05600	-18.73929	3.31671
2	2.53920	0	1.46913	1.46913	18.47576	-15.15906	3.31670
3	-4.30577	-6.84497	8.31409	1.46912	11.63079	-8.31409	3.31670
4	-7.96828	-9.23788	3.08024	-6.15764	0	-5.23386	-5.23386
CL	6.89220			6.89220			6.89221
ΔJL	0			-.25392			1.59365

Table 30

OVERBALANCE FACTORS FOR UNIFORM DISTRIBUTION OF EQUAL FORCES ON

END JOINTS OF A SEQUENCE

[$n = 4$; $k = 0.90$]

(1) Joint	(2) R_{me}	(3) ΣCO	(4) Net	(5) R_{fe}	(6) ΣCO	(7) Net
0	-4.35301	-0.81615	-5.16916	0	-4.73386	-4.73386
1	1.81367	1.63243	3.44610	10.51969	-6.20299	4.31670
2	5.07840	-1.63230	3.44610	-13.78442	-9.46772	4.31670
3	1.81367	1.63243	3.44610	10.51969	-6.20299	4.31670
4	-4.35301	-.81615	-5.16916	0	-4.73386	-4.73386
CL			6.89221			6.89221
ΔJL			0			.87060

Table 31

COMPUTATIONS FOR SHIFTING POINT OF APPLICATION OF CONCENTRATED LOAD

$[n = 4;\ k = 0.9]$

(1) Joint	(2) R_m	(3) γR_{B-1}	(4) ΔR_B	(5) R_2	(6) ΣCO	(7) Net
0	3.61527		−2.75110	0.86417	−0.27776	0.58641
1	6.11944		−5.50220	.61724	.55560	1.17284
2	2.53920	0	−5.50220	−2.96300	−.55553	−3.51853
3	−4.30577	10.42523	−5.50220	.61726	.55557	1.17283
4	−7.96828	11.58358	−2.75110	.86420	−.27777	.58643
Σ		22.00881				
IJL	1.72305			1.17284		1.17284
CL	6.89222			4.69137		4.69137

$$\gamma = 6.89222 \times \frac{2}{2.93826} = 4.69136$$

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Table 32

COMPUTATION OF RESISTANCE AT JOINT 2 FOR LIQUIDATION OF UNIFORMLY VARYING LOAD

ON EVEN NUMBER OF BAYS

[Transfer factor, 0.90]

Row	Item	Formula	$i = 2$	$i = 4$	$i = 6$	$i = 8$
1	i		2	4	6	8
2	$2i/k$	$2.22222\ (1)_i$	4.44444	8.88888	13.33333	17.77778
3	R_i R_2-term	$(5)_{i-2} - (3)_{i-4}$	1.00000	2.22222	3.93826	6.52946
4	W-term	$(6)_{i-2} + (2)_{i-2} - (4)_{i-4}$	0	4.44444	18.76540	50.58972
5	$2\ R_i/k\ R_2$-term	$2.22222\ (3)_i$	2.22222	4.93826	8.75168	14.50990
6	W-term	$2.22222\ (4)_i$	0	9.87652	41.70084	112.42148
7	$k\ R_i$ R_2-term	$0.90\ (3)_i$	0.90000	2.00000	3.54443	5.87651
8	W-term	$0.90\ (4)_i$	0	4.00000	16.88888	45.53074
9	$-kR_{i-2} + R_i + n$					
	R_2-term	$(3)_i - (7)_{i-2}$	1.00000	1.32222	1.93826	2.98503
10	W-term	$(4)_i - (8)_{i-2} + (1)_i$	2.00000	8.44444	20.76540	41.70086
11	R_2	$-(10)_i/(9)_i$	-2.00000	-6.38656	-10.71342	-13.97000

Table 33

COMPUTATION OF RESISTANCE AT JOINT 1 FOR LIQUIDATION OF UNIFORMLY VARYING LOAD

ON ODD NUMBER OF BAYS

[Transfer factor, 0.90]

Row	Item	Formula	$i = 1$	$i = 3$	$i = 5$	$i = 7$
1	i		1	3	5	7
2	$2i/k$	$2.22222\ (1)_i$	2.22222	6.66666	11.11111	15.55556
3	R_i R_1–term	$(5)_{i-2} - (3)_{i-4}$	1.00000	[a]3.22222	6.16049	10.46776
4	W–term	$(6)_{i-2} - (2)_{i-2} - (4)_{i-4}$	0	[a]2.22222	11.60492	34.67760
5	$2\ R_i/k$ R_1–term	$2.22222\ (3)_i$	2.22222	7.16049	13.68998	23.26169
6	W–term	$2.22222\ (4)_i$	0	4.93826	25.78871	77.06132
7	$k\ R_i$ R_1–term	$0.90\ (3)_i$	0.90000	2.90000	5.54444	9.42098
8	W–term	$0.90\ (4)_i$	0	2.00000	10.44443	31.20984
9	$-kR_{i-2} + R_i + n$ R_1–term	$(3)_i - (7)_{i-2}$	1.00000	2.32222	3.26049	4.92332
10	W–term	$(4)_i - (8)_{i-2} + (1)_i$	1.00000	5.22222	14.60492	31.23317
11	R_1	$-(10)_i/(9)_i$	−1.00000	−2.24881	−4.47936	−6.34392

[a]$R_3 = (1 + 2/k)\ R_1 + 2/k$.

Table 34

COMPUTATION AND VERIFICATION OF RESISTANCES FOR LIQUIDATION OF UNIFORMLY VARYING LOAD OVER EIGHT BAYS

[Transfer factor, 0.90; R_2 = −13.9700]

(1) Joint	(2) R_2–term	(3) R_1 W–term	(4) −13.9700 × R_2–term	(5) Net R_1	(6) ΣCO	(7) Net force
0	0	0	0	0	6.28650	6.28650
2	1.00000	0	−13.97000	−13.97000	11.96999	−2.00001
4	2.22222	4.44444	−31.04441	−26.59997	22.59994	−4.00003
6	3.93826	18.76540	−55.01749	−36.25209	30.25206	−6.00003
8_i	6.52946	50.58972	−91.21656	−40.62684		
8_n				−20.31342	16.31344	−3.99998

Table 35

COMPUTATION AND VERIFICATION OF RESISTANCES FOR LIQUIDATION OF

UNIFORMLY VARYING LOAD OVER SEVEN BAYS

[Transfer factor, 0.90; R_1 = −6.34392]

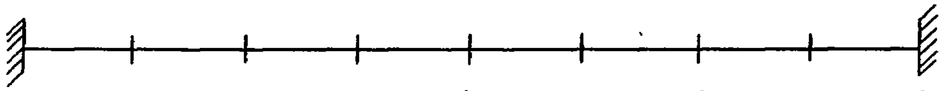
(1) Joint	(2) R_1–term	(3) R_1 W–term	(4) −6.34392 × R_1–term	(5) Net R_1	(6) ΣCO	(7) Net force
1	1.00000	0	−6.34392	−6.34392	5.34392	−1.00000
3	3.22222	2.22222	−20.44151	−18.21929	15.21929	−3.00000
5	6.16049	11.60492	−39.08166	−27.47674	22.47674	−5.00000
7_1	10.46776	34.67760	−66.49663	−31.72903		
7_n				−15.86451	12.36453	−3.49998

Table 36

COMPUTATION OF CORRECTIONS FOR ORIGINAL TABLE OF RESISTANCES FOR UNIFORM DISTRIBUTION OF A SINGLE FORCE APPLIED AT AN END JOINT WITH OTHER END ASSUMED FIXED

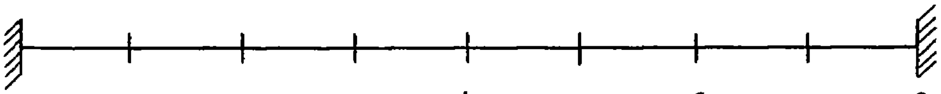
[Transfer factor, 0.88]

(1) Joint	(2) Original R_1	(3) ΣCO	(4) Error	(5) $\Delta CO1$	(6) $\Delta CO2$	(7) ΔR_1	(8) Corrected R_1	(9) ΣCO	(10) Error
0	0	0.50000	0	0	0	0	0	0.50000	0
1	-1.13637	2.13637	0	0	0	0	-1.13637	2.13637	0
2	-4.85539	5.85540	1×10^{-5}	-1	0	0	-4.85539	5.85539	0
3	-12.17136	13.17137	1	-3	0	2	-12.17134	13.17134	0
4	-25.07955	26.07958	3	-9	-1	7	-25.07948	26.07949	1×10^{-5}
5	-47.10042	48.10047	5	-22	-3	20	-47.10022	48.10022	0
6	-84.23971	85.23981	10	-51	-9	50	-84.23921	85.23922	1
7	-146.62643	147.62660	17	-111	-22	116	-146.62527	147.62528	1
8	-251.27530	252.27561	31	-232	-51	252	-251.27278	252.27258	0
9	-426.72722	427.72773	51	-467	-111	527	-426.72195	427.72195	0
10	-720.83318	721.83405	87	-916	-232	1061	-720.82257	721.82257	0



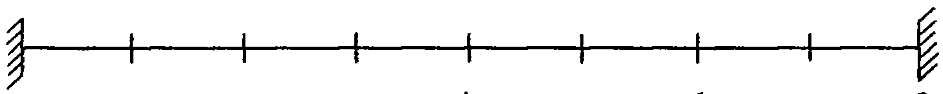
Joint	0	1	2	3	4	5	6	7	8
Original force	0	0	0	0	0	8	0	0	0
Final force	3	0	0	0	0	0	0	0	5

Figure 1.– Diagram showing points 0 to 8. Force to be transferred to joints 0 and 8 located at joint 5.



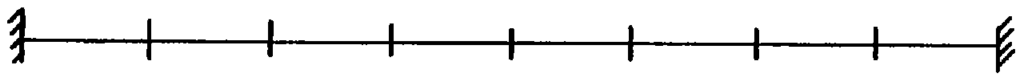
Joint	0	1	2	3	4	5	6	7	8
Original force	0	0	0	0	0	8	0	0	0
Final force	3	0	0	0	0	0	0	0	5
CO1	0	0	$-\frac{1}{2}R_1$	$-\frac{1}{2}R_2$	$-\frac{1}{2}R_3$	$-\frac{1}{2}R_4$	$-\frac{1}{2}R_5$	$-\frac{1}{2}R_6$	$-\frac{1}{2}R_7$
CO2	$-\frac{1}{2}R_1$	$-\frac{1}{2}R_2$	$-\frac{1}{2}R_3$	$-\frac{1}{2}R_4$	$-\frac{1}{2}R_5$	$-\frac{1}{2}R_6$	$-\frac{1}{2}R_7$	0	0
R	0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	0

Figure 2.– Overbalance diagram for transfer of single force to ends of sequence. Transfer factor of unity.



Joint	0	1	2	3	4	5	6	7	8
Original force	0	0	0	0	0	8	0	0	0
Final force	3	0	0	0	0	0	0	0	5
CO1	0	0	3	6	9	12	15	10	5
CO2	3	6	9	12	15	10	5	0	0
R	0	–6	–12	–18	–24	–30	–20	–10	0

Figure 3.– Overbalance diagram for transfer of single force to ends of sequence. Transfer factor of unity. Special case of general family of transfer patterns in which a characteristic load of $m + n$ pounds imposed at joint n of a sequence of $m + n$ bays is transferred to end joints 0 and $m + n$. $m = 3$; $n = 5$.



Joint	0	1	2	3	4	5	6	7	8
Original force	0	0	0	0	CL	0	0	0	0
Final force	1	0	0	0	0	0	0	0	1
CO1	0	0	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_1$
CO2	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_1$	0	0
R	0	R_1	R_2	R_3	R_4	R_3	R_2	R_1	0

Figure 4. – Overbalance diagram for transfer of single force from center of sequence to fixed end joints. Transfer factor other than unity.

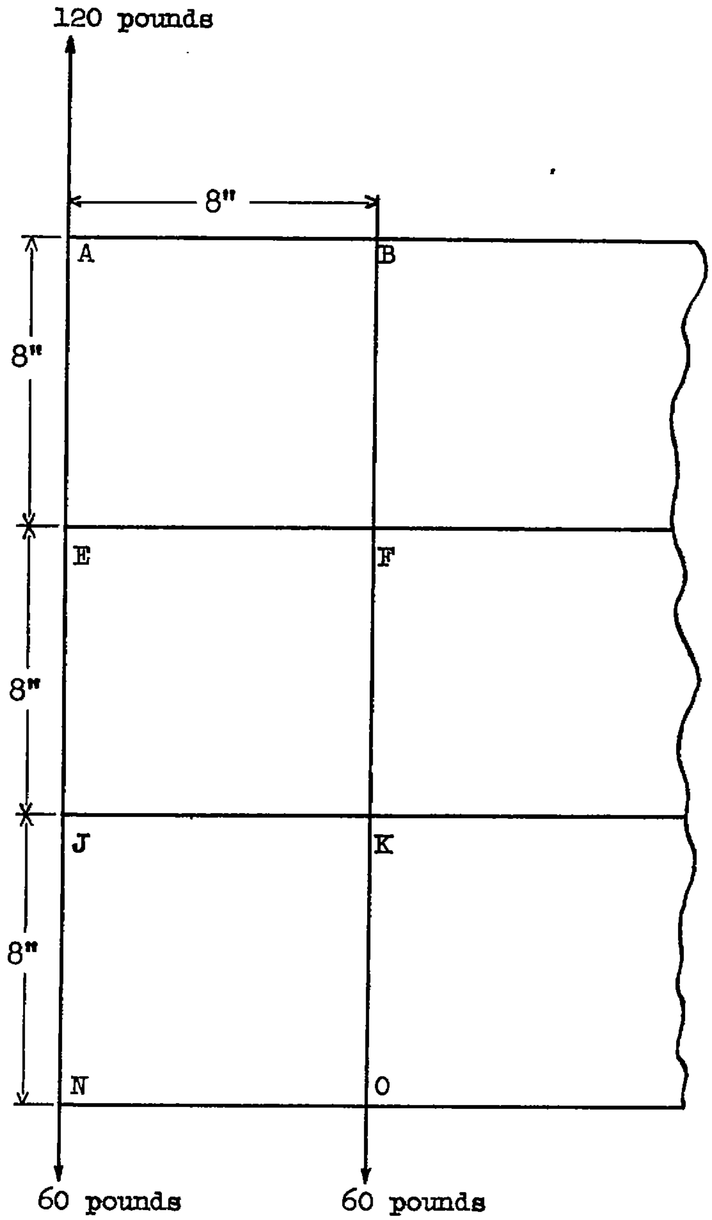


Figure 5. – Schematic drawing of one-half the flat panel.

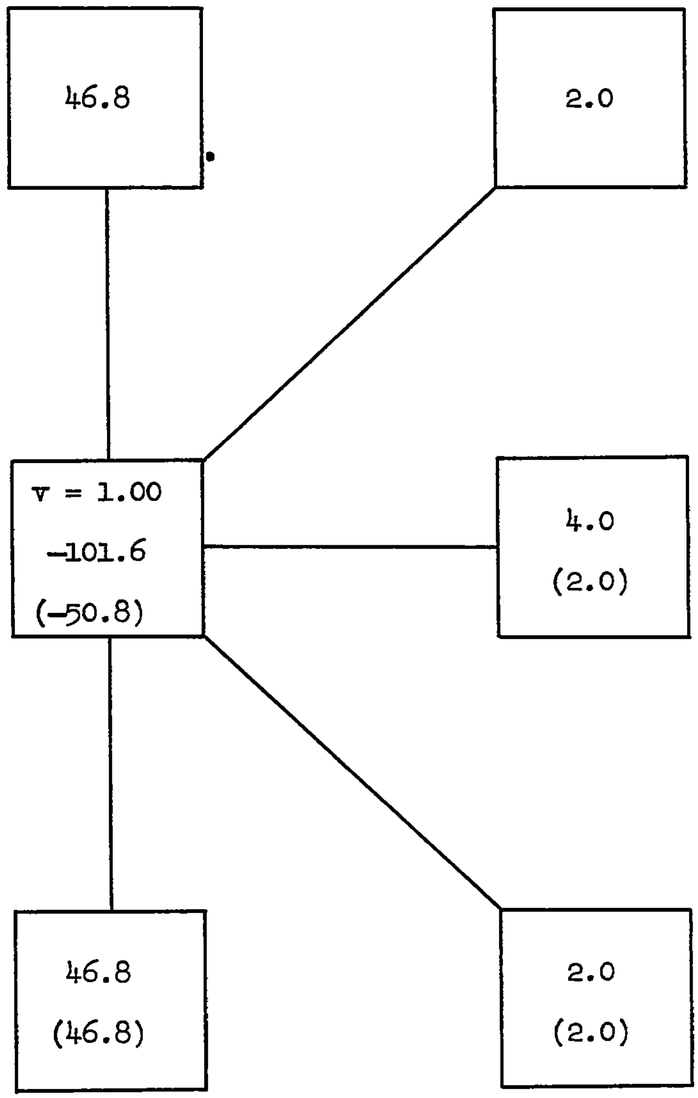


Figure 6. — Operations diagram for stringer AEJN. Unit block displacement of stringer develops resistance of −8.0 at each intermediate joint. k = 0.921.

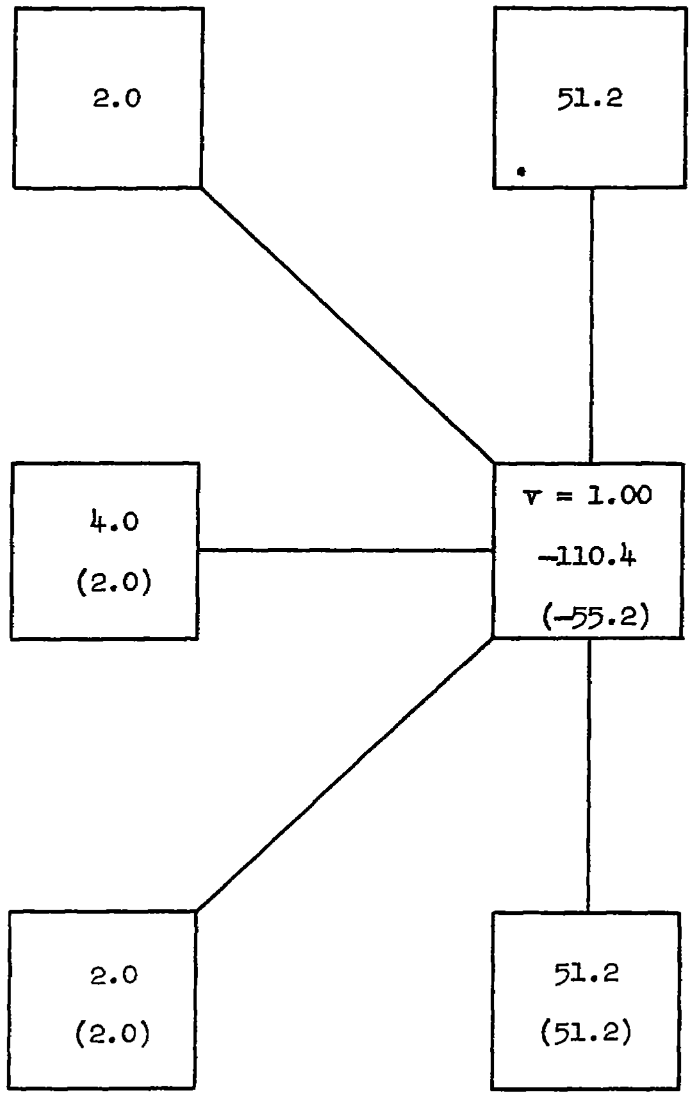
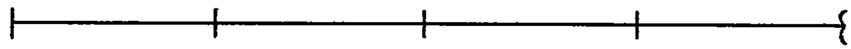
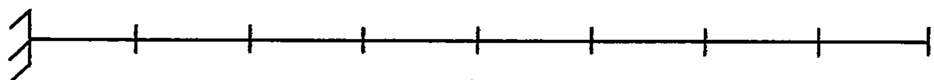


Figure 7. – Operations diagram for stringer BFKO. Unit block displacement of stringer develops resistance of –8.0 at each intermediate joint. k = 0.928.



Joint	0	1	2	3	4
Original force	0	0	0	0	W
Final force	0	0	0	0	0
CO1	0	$-kR_0$	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$
CO2	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_4$	– –
R	R_0	R_1	R_2	R_3	R_4

Figure 8.– Overbalance diagram for unloaded sector of a sequence.



Joint	0	1	2	3	4	5	6	7	8
Original force	0	0	0	0	0	0	0	0	CL
Final force	$\frac{1}{2}$	1	1	1	1	1	1	1	$\frac{1}{2}$
CO1	0	0	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_5$	$-\frac{k}{2}R_6$	$-\frac{k}{2}R_7$
CO2	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_5$	$-\frac{k}{2}R_6$	$-\frac{k}{2}R_7$	$-\frac{k}{2}R_8$	0
R	0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8

Figure 9.– Overbalance diagram for uniform distribution of single force at one end of a sequence. Other end fixed.

Joint	-4	-2	0	2	4	6	$n-4$	$n-2$	n
Original force	-4	-2	0	2	4	6	$n-4$	$n-2$	$n/2$
Final force	0	0	0	0	0	0	0	0	0
CO1	$\frac{k}{2}R_6$	$\frac{k}{2}R_4$	$\frac{k}{2}R_2$	0	$-\frac{k}{2}R_2$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_{n-6}$	$-\frac{k}{2}R_{n-4}$	$-\frac{k}{2}R_{n-2}$
CO2	$\frac{k}{2}R_2$	0	$\frac{k}{2}R_2$	$-\frac{k}{2}R_4$	$-\frac{k}{2}R_6$	$-\frac{k}{2}R_8$	$-\frac{k}{2}R_{n-2}$	$-k\ R_n$	0
R	$-R_4$	$-R_2$	0	R_2	R_4	R_6	R_{n-4}	R_{n-2}	R_n

Figure 10.— Overbalance diagram for liquidation of uniformly varying load. Even number of bays.

Joint	-3	-1	1	3	5	$n-6$	$n-4$	$n-2$	n
Original force	-3	-1	1	3	5	$n-6$	$n-4$	$n-2$	$n/2$
Final force	0	0	0	0	0	0	0	0	0
CO1	$\frac{k}{2}R_5$	$\frac{k}{2}R_3$	$\frac{k}{2}R_1$	$-\frac{k}{2}R_1$	$-\frac{k}{2}R_3$	$-\frac{k}{2}R_{n-8}$	$-\frac{k}{2}R_{n-6}$	$-\frac{k}{2}R_{n-4}$	$-\frac{k}{2}R_{n-2}$
CO2	$\frac{k}{2}R_1$	$\frac{k}{2}R_1$	$-\frac{k}{2}R_5$	$-\frac{k}{2}R_5$	$-\frac{k}{2}R_7$	$-\frac{k}{2}R_{n-4}$	$-\frac{k}{2}R_{n-2}$	$-kR_n$	0
R	$-R_3$	$-R_1$	R_1	R_3	R_5	R_{n-6}	R_{n-4}	R_{n-2}	R_n

Figure 11.— Overbalance diagram for liquidation of uniformly varying load. Odd number of bays.